UK Senior Mathematical Challenges

2007 to 2011

Organised by the

United Kingdom Mathematics Trust

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Contents

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RULES AND GUIDELINES (to be read before starting)

1. Do not open the question paper until the invigilator tells you to do so.
2. Detach the Answer Sheet (back page) and fill in your personal details before you open the question paper and begin.
   Once you have begun, record all your answers on the Answer Sheet.
3. Time allowed: **90 minutes**.
   No answers or personal details may be entered on the Answer Sheet after the 90 minutes are over.
4. The use of rough paper is allowed.
   Calculators, measuring instruments and squared paper are forbidden.
5. Candidates must be full-time students at secondary school or FE college, and must be in Year 13 or below (England & Wales); S6 or below (Scotland); Year 14 or below (Northern Ireland).
6. There are twenty-five questions. Each question is followed by five options marked *A*, *B*, *C*, *D*, *E*. Only one of these is correct. Enter the letter *A*-*E* corresponding to the correct answer in the corresponding box on the Answer Sheet.
7. **Scoring rules**: all candidates start out with 25 marks;
   0 marks are awarded for each question left unanswered;
   4 marks are awarded for each correct answer;
   1 mark is deducted for each incorrect answer.
8. **Guessing**: Remember that there is a penalty for wrong answers. Note also that later questions are deliberately intended to be harder than earlier questions. You are thus advised to concentrate first on solving as many as possible of the first 15-20 questions. Only then should you try later questions.

2007

1. What is the value of $\frac{2007}{9} + \frac{7002}{9}$?

A 500.5 B 545 C 1001 D 1655 E 2007

2. This morning Sam told Pat "I am getting married today, aged 30." From this information, Pat may correctly deduce that Sam was born in:

A 1976 or 1977 B 1977 C 1978 D 1979 E 1977 or 1978

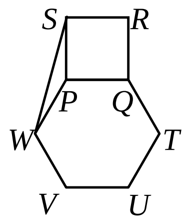
3. What is the value of $2006 \times 2008 - 2007 \times 2007$?

A -2007 B -1 C 0 D 1 E 4 026 042

4. The diagram shows square $PQRS$ and regular hexagon $PQTUVW$.

What is the size of $\angle PSW$?

A 10° B 12° C 15° D 24° E 30°



5. Which of the five expressions shown has a different value from the other four?

A 2^8 B 4^4 C $8^{8/3}$ D 16^2 E $32^{6/5}$

6. Cheryl finds a bag of coins. There are 50 coins inside and the value of the contents is £1.81. Given that it contains only two-pence and five-pence coins, how many more five-pence coins are there inside the bag than two-pence coins?

A 4 B 6 C 8 D 10 E 12

7. How many whole numbers between 1 and 2007 are divisible by 2 but not by 7?

A 857 B 858 C 859 D 860 E 861

8. Travelling at an average speed of 100 km/hr, a train took 3 hours to travel to Birmingham. Unfortunately the train then waited just outside the station, which reduced the average speed for the whole journey to 90 km/hr. For how many minutes was the train waiting?

A 1 B 5 C 10 D 15 E 20

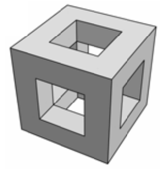
9. In a sale, a shopkeeper reduced the advertised selling price of a dress by 20%. This resulted in a profit of 4% over the cost price of the dress. What percentage profit would the shopkeeper have made if the dress had been sold at the original selling price?

A 16% B 20% C 24% D 25% E 30%

10. In 1954, a total of 6 527 mm of rain fell at Sprinkling Tarn and this set a UK record for annual rainfall. The tarn has a surface area of 23 450 m^2. Roughly how many million litres of water fell on Sprinkling Tarn in 1954?

A 15 B 150 C 1 500 D 15 000 E 150 000

11. A $4 \times 4 \times 4$ cube has three $2 \times 2 \times 4$ holes drilled symmetrically all the way through, as shown.
What is the surface area of the resulting solid?



A 192 B 144 C 136 D 120 E 96

12. How many two-digit numbers N have the property that the sum of N and the number formed by reversing the digits of N is a square?

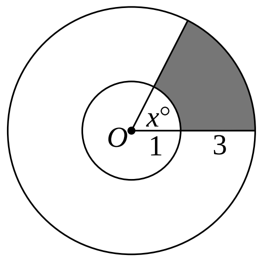
A 2 B 5 C 6 D 7 E 8

13. Which of the following gives the exact number of seconds in the last six complete weeks of 2007?

A 9! B 10! C 11! D 12! E 13!

{Note that $n! = 1 \times 2 \times 3 \times \ldots \times n$.*}*

14. The point O is the centre of both circles and the shaded area is one-sixth of the area of the outer circle.
What is the value of x?



A 60 B 64 C 72 D 80 E 84

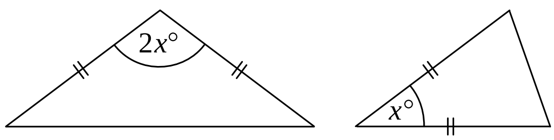
15. How many hexagons can be found in the diagram on the right if each side of a hexagon must consist of all or part of one of the straight lines in the diagram?

A 4 B 8 C 12 D 16 E 20

16. Damien wishes to find out if 457 is a prime number. In order to do this he needs to check whether it is exactly divisible by some prime numbers. What is the smallest number of possible prime number divisors that Damien needs to check before he can be sure that 457 is a prime number?

A 8 B 9 C 10 D 11 E 12

17. The two triangles have equal areas and the four marked lengths are equal.
What is the value of x?

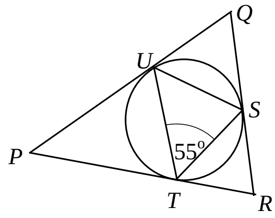


A 30 B 45 C 60 D 75 E more information needed

18. The year 1789 (when the French Revolution started) has three and no more than three adjacent digits (7, 8 and 9) which are consecutive integers in increasing order. How many years between 1000 and 9999 have this property?

A 130 B 142 C 151 D 169 E 180

19. The largest circle which it is possible to draw inside triangle PQR touches the triangle at S, T and U, as shown in the diagram. The size of $\angle STU = 55°$. What is the size of $\angle PQR$?



A 55° B 60° C 65° D 70° E 75°

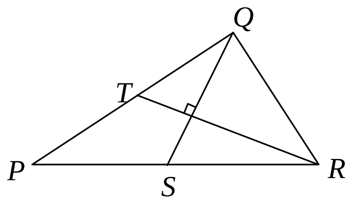
20. A triangle is cut from the corner of a rectangle. The resulting pentagon has sides of length 8, 10, 13, 15 and 20 units, though not necessarily in that order. What is the area of the pentagon?

A 252.5 B 260 C 270 D 275.5 E 282.5

21. A bracelet is to be made by threading four identical red beads and four identical yellow beads onto a hoop. How many different bracelets can be made?

A 4 B 8 C 12 D 18 E 24

22. In triangle PQR, S and T are the midpoints of PR and PQ respectively; QS is perpendicular to RT; $QS = 8$; $RT = 12$. What is the area of triangle PQR?

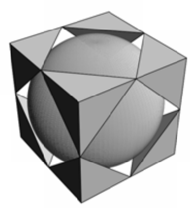


A 24 B 32 C 48 D 64 E 96

23. The sum of the lengths of the 12 edges of a cuboid is x cm. The distance from one corner of the cuboid to the furthest corner is y cm. What, in cm^2, is the total surface area of the cuboid?

A $\frac{x \quad 2y}{2}$ B $x^2 + y^2$ C $\frac{x \quad 4y}{4}$ D $\frac{xy}{6}$ E $\frac{x \quad 16y}{16}$

24. A paperweight is made from a glass cube of side 2 units by first shearing off the eight tetrahedral corners which touch at the midpoints of the edges of the cube. The remaining inner core of the cube is discarded and replaced by a sphere. The eight corner pieces are now stuck onto the sphere so that they have the same positions relative to each other as they did originally. What is the diameter of the sphere?



A $\sqrt{8} - 1$ B $\sqrt{8} + 1$ C $\frac{1}{3}(6 + \sqrt{3})$ D $\frac{4}{3}\sqrt{3}$ E $2\sqrt{3}$

25. The line with equation $y = x$ is an axis of symmetry of the curve with equation

$$y = \frac{px + q}{rx + s},$$

where p, q, r, s are all non-zero. Which of the following is necessarily true?

A $p + q = 0$ B $r + s = 0$ C $p + r = 0$ D $p + s = 0$ E $q + r = 0$

2008

1. What is the value of $2 \times 2008 + 2008 \times 8$?

A 4016 B 16064 C 20080 D 64256 E 80020

2. A giant thresher shark weighing 1250 pounds, believed to be the heaviest ever caught, was landed by fisherman Roger Nowell off the Cornish coast in November 2007. The fish was sold by auction at Newlyn Fish Market for £255. Roughly, what was the cost per pound?

A 5p B 20p C 50p D £2 E £5

3. What is the value of $\sqrt{\frac{1}{2^6} + \frac{1}{6^2}}$?

A $\frac{1}{10}$ B $\frac{1}{9}$ C $\frac{1}{3}$ D $\frac{5}{24}$ E $\frac{7}{24}$

4. In this subtraction, P, Q, R and S are digits. What is the value of $P + Q + R + S$?

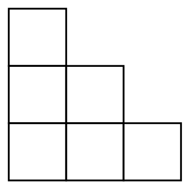
$$\begin{array}{r} 8\ Q\ 0\ S \\ -\ P\ 0\ R\ 2 \\ \hline 2\ 0\ 0\ 8 \end{array}$$

A 12 B 14 C 16 D 18 E 20

5. 200 T-shirts have been bought for a Fun Run at a cost of £400 plus VAT at $17\frac{1}{2}$%. The cost of entry for the run is £5 per person. What is the minimum number of entries needed in order to cover the total cost of the T-shirts?

A 40 B 47 C 80 D 84 E 94

6. It is required to shade at least one of the six small squares in the diagram on the right so that the resulting figure has exactly one axis of symmetry. In how many different ways can this be done?



A 6 B 9 C 10 D 12 E 15

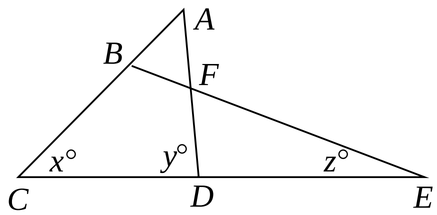
7. A newspaper headline read ‘Welsh tortoise recaptured 1.8 miles from home after 8 months on the run’. Assuming the tortoise travelled in a straight line, roughly how many minutes did the tortoise take on average to ‘run’ one foot?

[1 mile = 5280 feet]

A 3 B 9 C 16 D 36 E 60

8. In the figure shown, $AB = AF$ and ABC, AFD, BFE and CDE are all straight lines.

Which of the following expressions gives z in terms of x and y?



A $\frac{y - x}{2}$ B $y - \frac{x}{2}$ C $\frac{y - x}{3}$ D $y - \frac{x}{3}$ E $y - x$

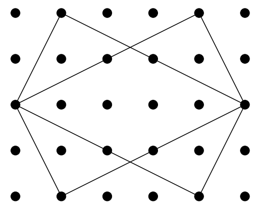
9. What is the remainder when the 2008-digit number 222 ... 22 is divided by 9?

A 8 B 6 C 4 D 2 E 0

10. Which one of the following rational numbers *cannot* be expressed as $\frac{1}{m} + \frac{1}{n}$ where m, n are different positive integers?

A $\frac{3}{4}$ B $\frac{3}{5}$ C $\frac{3}{6}$ D $\frac{3}{7}$ E $\frac{3}{8}$

11. The distance between two neighbouring dots in the dot lattice is 1 unit. What, in square units, is the area of the region where the two rectangles overlap?



A 6 B $6\frac{1}{4}$ C $6\frac{1}{2}$ D 7 E $7\frac{1}{2}$

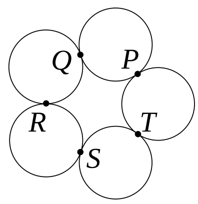
12. Mr and Mrs Stevens were married on a Saturday in July 1948. On what day of the week did their diamond wedding anniversary fall this year?

A Monday B Tuesday C Thursday D Friday E Saturday

13. Positive integers m and n are such that $2^m + 2^n = 1280$. What is the value of $m + n$?

A 14 B 16 C 18 D 32 E 640

14. Five touching circles each have radius 1 and their centres are at the vertices of a regular pentagon. What is the radius of the circle through the points of contact P, Q, R, S and T?



A $\tan 18°$ B $\tan 36°$ C $\tan 45°$ D $\tan 54°$ E $\tan 72°$

15. A sequence of positive integers t_1, t_2, t_3, t_4, … is defined by:

$$t_1 = 13; \; t_{n+1} = \tfrac{1}{2}t_n \text{ if } t_n \text{ is even}; \; t_{n+1} = 3t_n + 1 \text{ if } t_n \text{ is odd.}$$

What is the value of t_{2008}?

A 1 B 2 C 4 D 8 E None of these.

16. The numbers x, y and z satisfy the equations

$$x + y + 2z = 850, \qquad x + 2y + z = 950, \qquad 2x + y + z = 1200.$$

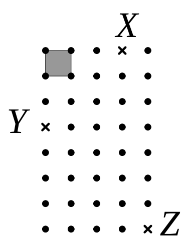
What is their mean?

A 250 B $\frac{1000}{3}$ C 750 D 1000 E More information is needed.

17. Andy and his younger cousin Alice both have their birthdays today. Remarkably, Andy is now the same age as the sum of the digits of the year of his birth and the same is true of Alice. How many years older than Alice is Andy?

A 10 B 12 C 14 D 16 E 18

18. The shaded square of the lattice shown has area 1. What is the area of the circle through the points X, Y and Z?

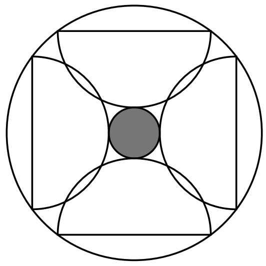


A $\frac{9\pi}{2}$ B 8π C $\frac{25\pi}{2}$ D 25π E 50π

19. How many prime numbers p are there such that $199p + 1$ is a perfect square?

A 0 B 1 C 2 D 4 E 8

20. The diagram shows four semicircles symmetrically placed between two circles. The shaded circle has area 4 and each semicircle has area 18. What is the area of the outer circle?



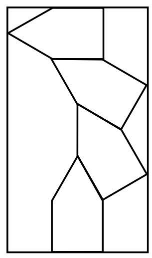
A $72\sqrt{2}$ B 100 C 98 D 96 E $32\sqrt{3}$

21. The fraction $\frac{2008}{1998}$ may be written in the form $a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}}$ where a, b, c and d are positive integers. What is the value of d?

A 2 B 4 C 5 D 199 E 1998

22. A pentagon is made by attaching an equilateral triangle to a square with the same edge length. Four such pentagons are placed inside a rectangle, as shown.

What is the ratio of the length of the rectangle to its width?



A $\sqrt{3}:1$ B $2:1$ C $\sqrt{2}:1$ D $3:2$ E $4:\sqrt{3}$

23. How many pairs of real numbers (x, y) satisfy the equation $(x + y)^2 = (x + 3)(y - 3)$?

A 0 B 1 C 2 D 4 E infinitely many

24. The length of the hypotenuse of a particular right-angled triangle is given by $\sqrt{1 + 3 + 5 + 7 + \ \dots \ + 25}$. The lengths of the other two sides are given by $\sqrt{1 + 3 + 5 + \ \dots \ + x}$ and $\sqrt{1 + 3 + 5 + \ \dots \ + y}$ where x and y are positive integers. What is the value of $x + y$?

A 12 B 17 C 24 D 28 E 32

25. What is the area of the polygon formed by all points (x, y) in the plane satisfying the inequality $\left||x| - 2\right| + \left||y| - 2\right| \leqslant 4$?

A 24 B 32 C 64 D 96 E 112

2009

1. What is 20% of 30%?

A 6% B 10% C 15% D 50% E 60%

2. Which of the following is not a multiple of 15?

A 135 B 315 C 555 D 785 E 915

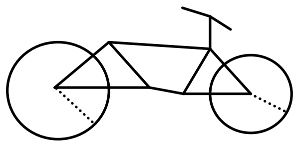
3. What is the value of $1^6 - 2^5 + 3^4 - 4^3 + 5^2 - 6^1$?

A 1 B 2 C 3 D 4 E 5

4. Steve travelled 150 miles on a motorbike and used 10 litres of petrol. Given that 1 gallon ≈ 4.5 litres, roughly how many miles per gallon did Steve achieve on his journey?

A 10 B 20 C 40 D 50 E 70

5. Boris Biker entered the Tour de Transylvania with an unusual bicycle whose back wheel is larger than the front. The radius of the back wheel is 40 cm, and the radius of the front wheel is 30 cm. On the first stage of the race the smaller wheel made 120000 revolutions. How many revolutions did the larger wheel make?



A 90000 B 90000π C 160000 D $\dfrac{160000}{\pi}$ E 120000

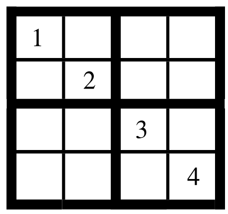
6. A bag contains hundreds of glass marbles, each one coloured either red, orange, green or blue. There are more than 2 marbles of each colour.

Marbles are drawn randomly from the bag, one at a time, and not replaced.

How many marbles must be drawn from the bag in order to ensure at least three marbles of the same colour are drawn?

A 4 B 7 C 9 D 12 E 13

7. A mini-sudoku is a 4 by 4 grid, where each row, column and 2 by 2 outlined block contains the digits 1, 2, 3 and 4 once and once only. How many different ways are there of completing the mini-sudoku shown?

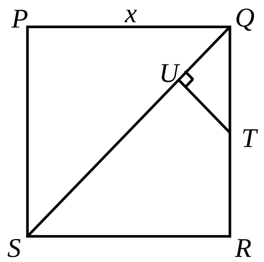


A 1 B 2 C 4 D 8 E 12

8. The entries to the Senior Mathematical Challenge grew from 87400 in 2007 to 92690 in 2008. Approximately what percentage increase does this represent?

A 4% B 5% C 6% D 7% E 8%

9. A square $PQRS$ has sides of length x. T is the midpoint of QR and U is the foot of the perpendicular from T to QS. What is the length of TU?



A $\frac{x}{2}$ B $\frac{x}{3}$ C $\frac{x}{\sqrt{2}}$ D $\frac{x}{2\sqrt{2}}$ E $\frac{x}{4}$

10. Consider all three-digit numbers formed by using *different* digits from 0, 1, 2, 3 and 5. How many of these numbers are divisible by 6?

A 4 B 7 C 10 D 15 E 20

11. For what value of x is $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 2^x$ true?

A $\frac{1}{2}$ B $1\frac{1}{2}$ C $2\frac{1}{2}$ D $3\frac{1}{2}$ E $4\frac{1}{2}$

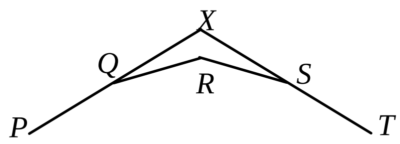
12. Which of the following has the greatest value?

A $\cos 50°$ B $\sin 50°$ C $\tan 50°$ D $\frac{1}{\sin 50°}$ E $\frac{1}{\cos 50°}$

13. Suppose that $x - \frac{1}{x} = y - \frac{1}{y}$ and $x \neq y$. What is the value of xy?

A 4 B 1 C −1 D −4 E more information is needed

14. P, Q, R, S, T are vertices of a regular polygon. The sides PQ and TS are produced to meet at X, as shown in the diagram, and $\angle QXS = 140°$. How many sides does the polygon have?



A 9 B 18 C 24 D 27 E 40

15. For how many integers n is $\frac{n}{100 - n}$ also an integer?

A 1 B 6 C 10 D 18 E 100

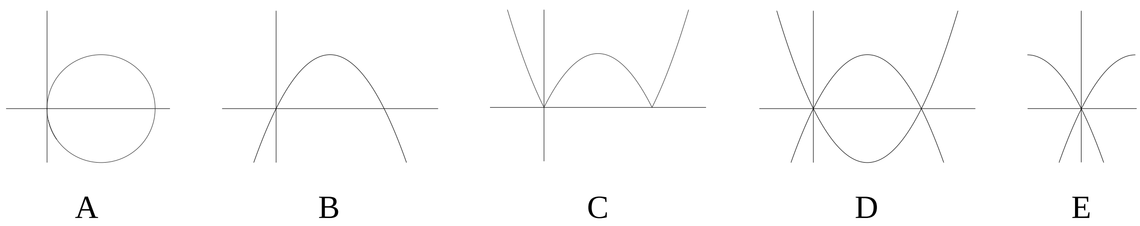
16. The positive numbers x and y satisfy the equations $x^4 - y^4 = 2009$ and $x^2 + y^2 = 49$. What is the value of y?

A 1 B 2 C 3 D 4 E more information is needed

17. A solid cube is divided into two pieces by a single rectangular cut. As a result, the total surface area increases by a fraction f of the surface area of the original cube. What is the greatest possible value of f?

A $\frac{1}{3}$ B $\frac{\sqrt{3}}{4}$ C $\frac{\sqrt{2}}{3}$ D $\frac{1}{2}$ E $\frac{1}{\sqrt{3}}$

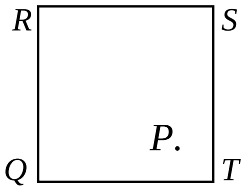
18. Which of the following could be part of the graph of the curve $y^2 = x(2 - x)$?



19. Hamish and his friend Ben live in villages which are 51 miles apart. During the summer holidays, they agreed to cycle towards each other along the same main road. Starting at noon, Hamish cycled at x mph. Starting at 2 pm, Ben cycled at y mph. They met at 4 pm. If they had both started at noon, they would have met at 2.50 pm. What is the value of y?

A 7.5 B 8 C 10.5 D 12 E 12.75

20. A point P is chosen at random inside a square $QRST$. What is the probability that $\angle RPQ$ is acute?



A $\frac{3}{4}$ B $\sqrt{2}-1$ C $\frac{1}{2}$ D $\frac{\pi}{4}$ E $1-\frac{\pi}{8}$

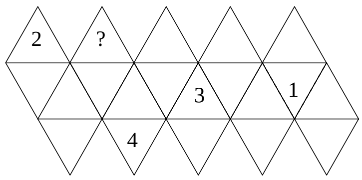
21. A frustum is the solid obtained by slicing a right-circular cone perpendicular to its axis and removing the small cone above the slice. This leaves a shape with two circular faces and a curved surface. The original cone has base radius 6 cm and height 8 cm, and the curved surface area of the frustum is equal to the area of the two circles. What is the height of the frustum?

A 3 cm B 4 cm C 5 cm D 6 cm E 7 cm

22. M and N are the midpoints of sides GH and FG, respectively, of parallelogram $EFGH$. The area of triangle ENM is 12 cm^2. What is the area of the parallelogram $EFGH$?

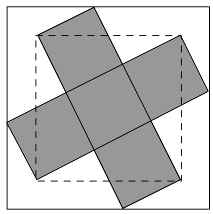
A 20 cm^2 B 24 cm^2 C 32 cm^2 D 48 cm^2 E more information is required

23. The net shown is folded into an icosahedron and the remaining faces are numbered such that at each vertex the numbers 1 to 5 all appear. What number must go on the face with a question mark?



A 1 B 2 C 3 D 4 E 5

24. A figure in the shape of a cross is made from five 1×1 squares, as shown. The cross is inscribed in a large square whose sides are parallel to the dashed square, formed by four of the vertices of the cross. What is the area of the large outer square?



A 9 B $\frac{49}{5}$ C 10 D $\frac{81}{8}$ E $\frac{32}{3}$

25. Four positive integers a, b, c and d are such that

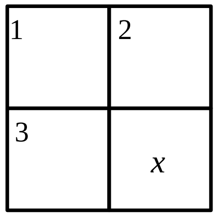
$$abcd + abc + bcd + cda + dab + ab + bc + cd + da + ac + bd + a + b + c + d = 2009.$$

What is the value of $a + b + c + d$?

A 73 B 75 C 77 D 79 E 81

2010

1. What is the digit x in this cross-number?



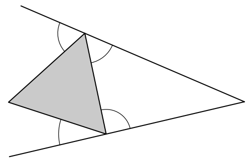
Across	Down
1. A cube	1. One less than a cube
3. A cube	

A 2 B 3 C 4 D 5 E 6

2. What is the smallest possible value of $20p + 10q + r$ when p, q and r are *different* positive integers?

A 31 B 43 C 53 D 63 E 2010

3. The diagram shows an equilateral triangle touching two straight lines. What is the sum of the four marked angles?



A 120° B 180° C 240° D 300° E 360°

4. The year 2010 is one in which the sum of its digits is a factor of the year itself. How many more years will it be before this is next the case?

A 3 B 6 C 9 D 12 E 15

5. A notice on Morecambe promenade reads: 'It would take 20 million years to fill Morecambe Bay from a bath tap.' Assuming that the flow from the bath tap is 6 litres a minute, what does the notice imply is the approximate capacity of Morecambe Bay in litres?

A 6×10^{10} B 6×10^{11} C 6×10^{12} D 6×10^{13} E 6×10^{14}

6. Dean runs up a mountain road at 8 km per hour. It takes him one hour to get to the top. He runs down the same road at 12 km per hour. How many minutes does it take him to run down the mountain?

A 30 B 40 C 45 D 50 E 90

7. There are 120 different arrangements of the five letters in the word ANGLE. If all 120 are listed in alphabetical order starting with AEGLN and finishing with NLGEA, which position in the list does ANGLE occupy?

A 18th B 20th C 22nd D 24th E 26th

8. Which of the following is equivalent to $(x + y + z)(x - y - z)$?

A $x^2 - y^2 - z^2$ B $x^2 - y^2 + z^2$ C $x^2 - xy - xz - z^2$ D $x^2 - (y + z)^2$ E $x^2 - (y - z)^2$

9. The symbol ◊ is defined by $x \lozenge y = x^y - y^x$. What is the value of $(2 \lozenge 3) \lozenge 4$?

A -3 B $-\frac{3}{4}$ C 0 D $\frac{3}{4}$ E 3

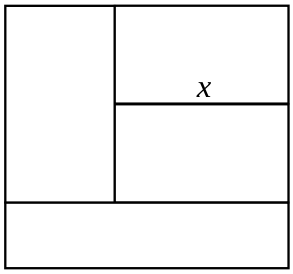
10. A square is cut into 37 squares of which 36 have area 1 cm^2. What is the length of the side of the *original* square?

A 6 cm　　B 7 cm　　C 8 cm　　D 9 cm　　E 10 cm

11. What is the median of the following numbers?

A $9\sqrt{2}$　　B $3\sqrt{19}$　　C $4\sqrt{11}$　　D $5\sqrt{7}$　　E $6\sqrt{5}$

12. The diagram, which is not to scale, shows a square with side length 1, divided into four rectangles whose areas are equal. What is the length labelled x?

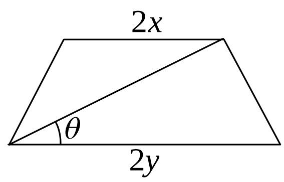


A $\frac{2}{3}$　　B $\frac{17}{24}$　　C $\frac{4}{5}$　　D $\frac{49}{60}$　　E $\frac{5}{6}$

13. How many two-digit numbers have remainder 1 when divided by 3 and remainder 2 when divided by 4?

A 8　　B 7　　C 6　　D 5　　E 4

14. The parallel sides of a trapezium have lengths $2x$ and $2y$ respectively. The diagonals are equal in length, and a diagonal makes an angle θ with the parallel sides, as shown. What is the length of each diagonal?



A $x + y$　　B $\dfrac{x + y}{\sin\theta}$　　C $(x+y)\cos\theta$　　D $(x+y)\tan\theta$　　E $\dfrac{x + y}{\cos\theta}$

15. What is the smallest prime number that is equal to the sum of two prime numbers and is also equal to the sum of three different prime numbers?

A 7　　B 11　　C 13　　D 17　　E 19

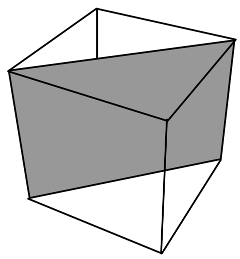
16. *PQRS* is a quadrilateral inscribed in a circle of which *PR* is a diameter. The lengths of *PQ*, *QR* and *RS* are 60, 25 and 52 respectively. What is the length of *SP*?

A $21\frac{2}{3}$　　B $28\frac{11}{13}$　　C 33　　D 36　　E 39

17. One of the following is equal to $\sqrt{9^{16x^2}}$ for all values of x. Which one?

A 3^{4x}　　B 3^{4x^2}　　C 3^{8x^2}　　D 9^{4x}　　E 9^{8x^2}

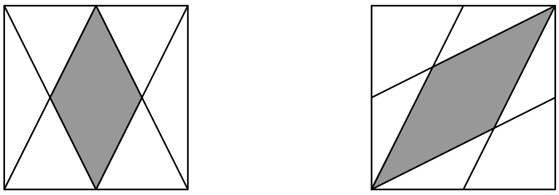
18. A solid cube of side 2 cm is cut into two triangular prisms by a plane passing through four vertices, as shown. What is the total surface area of these two prisms?



A $8(3 + \sqrt{2})$　　B $2(8 + \sqrt{2})$　　C $8(3 + 2\sqrt{2})$

D $16(3 + \sqrt{2})$　　E $8\sqrt{2}$

19. The diagrams show two different shaded rhombuses, each inside a square with sides of length 6.



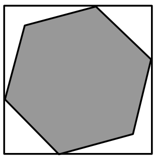
Each rhombus is formed by joining vertices of the square to midpoints of the sides of the square. What is the difference between the shaded areas?

A 4 B 3 C 2 D 1 E 0

20. There are 10 girls in a mixed class. If two pupils from the class are selected at random to represent the class on the School Council, then the probability that both are girls is 0.15. How many boys are in the class?

A 10 B 12 C 15 D 18 E 20

21. The diagram shows a regular hexagon, with sides of length 1, inside a square. Two vertices of the hexagon lie on a diagonal of the square and the other four lie on the edges.



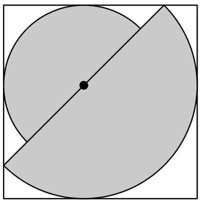
What is the area of the square?

A $2 + \sqrt{3}$ B 4 C $3 + \sqrt{2}$ D $1 + \frac{3\sqrt{3}}{2}$ E $\frac{7}{2}$

22. If $x^2 - px - q = 0$, where p and q are positive integers, which of the following could not equal x^3?

A $4x + 3$ B $8x + 5$ C $8x + 7$ D $10x + 3$ E $26x + 5$

23. The diagram shows two different semicircles inside a square with sides of length 2. The common centre of the semicircles lies on a diagonal of the square.



What is the total shaded area?

A π B $6\pi(3 - 2\sqrt{2})$ C $\pi\sqrt{2}$ D $3\pi(2 - \sqrt{2})$ E $8\pi(2\sqrt{2} - 3)$

24. Three spheres of radius 1 are placed on a horizontal table and inside a vertical hollow cylinder of height 2 units which is just large enough to surround them. What fraction of the internal volume of the cylinder is occupied by the spheres?

A $\frac{2}{7 + 4\sqrt{3}}$ B $\frac{2}{2 + \sqrt{3}}$ C $\frac{1}{3}$ D $\frac{3}{2 + \sqrt{3}}$ E $\frac{6}{7 + 4\sqrt{3}}$

25. All the digits of a number are different, the first digit is not zero, and the sum of the digits is 36. There are $N \times 7!$ such numbers. What is the value of N?

A 72 B 97 C 104 D 107 E 128

2011

1. Which of the numbers below is not a whole number?

A $\frac{2011+0}{1}$ B $\frac{2011+1}{2}$ C $\frac{2011+2}{3}$ D $\frac{2011+3}{4}$ E $\frac{2011+4}{5}$

2. Jack and Jill went up the hill to fetch a pail of water. Having filled the pail to the full, Jack fell down, spilling $\frac{2}{3}$ of the water, before Jill caught the pail. She then tumbled down the hill, spilling $\frac{2}{5}$ of the remainder.

What fraction of the pail does the remaining water fill?

A $\frac{11}{15}$ B $\frac{1}{3}$ C $\frac{4}{15}$ D $\frac{1}{5}$ E $\frac{1}{15}$

3. The robot *Lumber9* moves along the number line. *Lumber9* starts at 0, takes 1 step forward (to 1), then 2 steps backward (to −1), then 3 steps forward, 4 steps backward, and so on, moving alternately forwards and backwards, one more step each time. At what number is *Lumber9* after 2011 steps in total?

A 1006 B 27 C 11 D 0 E −18

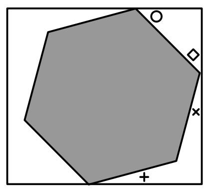
4. What is the last digit of 3^{2011}?

A 1 B 3 C 5 D 7 E 9

5. The diagram shows a regular hexagon inside a rectangle.

What is the sum of the four marked angles?

A 90° B 120° C 150° D 180° E 210°



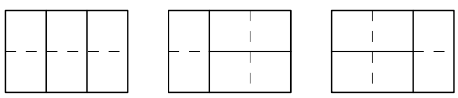
6. Granny and her granddaughter Gill both had their birthday yesterday. Today, Granny's age in years is an even number and 15 times that of Gill. In 4 years' time Granny's age in years will be the square of Gill's age in years. How many years older than Gill is Granny today?

A 42 B 49 C 56 D 60 E 64

7. Two sides of a triangle have lengths 4 cm and 5 cm. The third side has length x cm, where x is a positive integer. How many different values can x have?

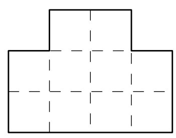
A 4 B 5 C 6 D 7 E 8

8. A 2 × 3 grid of squares can be divided into 1 × 2 rectangles in three different ways.



How many ways are there of dividing the bottom shape into 1 × 2 rectangles?

A 1 B 4 C 6 D 7 E 8



9. Sam has a large collection of 1 × 1 × 1 cubes, each of which is either red or yellow. Sam makes a 3 × 3 × 3 block from twenty-seven cubes, so that no cubes of the same colour meet face-to-face.

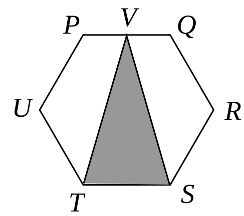
What is the difference between the largest number of red cubes that Sam can use and the smallest number?

A 0 B 1 C 2 D 3 E 4

10. A triangle has two edges of length 5. What length should be chosen for the third side of the triangle so as to maximise the area within the triangle?

A 5 B 6 C $5\sqrt{2}$ D 8 E $5\sqrt{3}$

11. $PQRSTU$ is a regular hexagon and V is the midpoint of PQ. What fraction of the area of $PQRSTU$ is the area of triangle STV?

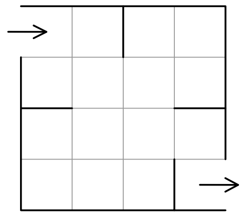


A $\frac{1}{4}$ B $\frac{2}{15}$ C $\frac{1}{3}$ D $\frac{2}{5}$ E $\frac{5}{12}$

12. The *primorial* of a number is the product of all of the prime numbers less than or equal to that number. For example, the primorial of 6 is $2 \times 3 \times 5 = 30$. How many different whole numbers have a primorial of 210?

A 1 B 2 C 3 D 4 E 5

13. The diagram represents a maze. Given that you can only move horizontally and vertically and are not allowed to revisit a square, how many different routes are there through the maze?



A 16 B 12 C 10 D 8 E 6

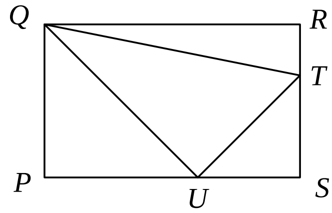
14. An equilateral triangle of side length 4 cm is divided into smaller equilateral triangles, all of which have side length equal to a whole number of centimetres. Which of the following cannot be the number of smaller triangles obtained?

A 4 B 8 C 12 D 13 E 16

15. The equation $x^2 + ax + b = 0$, where a and b are different, has solutions $x = a$ and $x = b$. How many such equations are there?

A 0 B 1 C 3 D 4 E an infinity

16. $PQRS$ is a rectangle. The area of triangle QRT is $\frac{1}{5}$ of the area of $PQRS$, and the area of triangle TSU is $\frac{1}{8}$ of the area of $PQRS$. What fraction of the area of rectangle $PQRS$ is the area of triangle QTU?



A $\frac{27}{40}$ B $\frac{21}{40}$ C $\frac{1}{2}$ D $\frac{19}{40}$ E $\frac{23}{60}$

17. Jamie conducted a survey on the food preferences of pupils at a school and discovered that 70% of the pupils like pears, 75% like oranges, 80% like bananas and 85% like apples. What is the smallest possible percentage of pupils who like all four of these fruits?

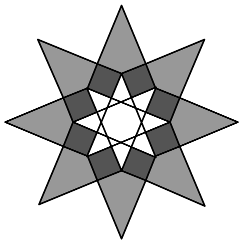
A at least 10% B at least 15% C at least 20%

D at least 25% E at least 70%

18. Two numbers x and y are such that $x + y = 20$ and $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$. What is the value of $x^2y + xy^2$?

A 80 B 200 C 400 D 640 E 800

19. The diagram shows a small regular octagram (an eight-sided star) surrounded by eight squares (dark grey) and eight kites (light grey) to make a large regular octagram. Each square has area 1.



What is the area of one of the light grey kites?

A 2 B $\sqrt{2}+1$ C $\frac{21}{8}$ D $4\sqrt{2}-3$ E $\frac{11}{4}$

20. Positive integers x and y satisfy the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$.
What is the maximum possible value of $\frac{x}{y}$?

A 2 B 4 C 8 D 11 E 44

21. Each of the Four Musketeers made a statement about the four of them, as follows.

d'Artagnan: "Exactly one is lying."

Athos: "Exactly two of us are lying."

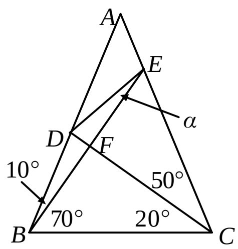
Porthos: "An odd number of us is lying."

Aramis: "An even number of us is lying."

How many of them were lying (with the others telling the truth)?

A one B one or two C two or three D three E four

22. In the diagram, $\angle ABE = 10°$; $\angle EBC = 70°$; $\angle ACD = 50°$; $\angle DCB = 20°$; $\angle DEF = \alpha$.



Which of the following is equal to $\tan \alpha$?

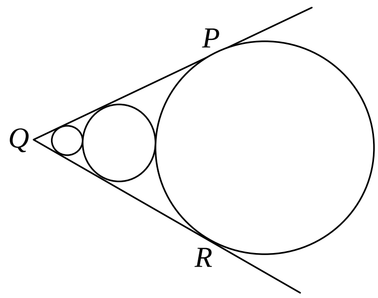
A $\frac{\tan 10° \tan 20°}{\tan 50°}$ B $\frac{\tan 10° \tan 20°}{\tan 70°}$ C $\frac{\tan 10° \tan 50°}{\tan 70°}$

D $\frac{\tan 20° \tan 50°}{\tan 70°}$ E $\frac{\tan 10° \tan 70°}{\tan 50°}$

23. What is the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$?

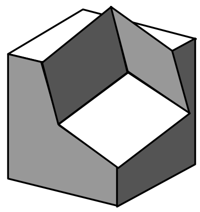
A -7 B -5 C -4 D -1 E 4

24. Three circles and the lines PQ and QR touch as shown. The distance between the centres of the smallest and the biggest circles is 16 times the radius of the smallest circle. What is the size of $\angle PQR$?



A 45° B 60° C 75° D 90° E 135°

25. A solid sculpture consists of a $4 \times 4 \times 4$ cube with a $3 \times 3 \times 3$ cube sticking out, as shown. Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each.



What is the total volume of the sculpture?

A 79 B 81 C 82 D 84 E 85

Summary of answers

	2007	2008	2009	2010	2011
1.	C	C	A	C	D
2.	A	B	D	B	D
3.	B	D	E	C	B
4.	C	C	E	B	D
5.	E	E	A	D	B
6.	A	E	C	B	C
7.	D	D	B	C	D
8.	E	A	C	D	C
9.	E	D	D	D	B
10.	B	D	B	E	C
11.	D	B	C	D	C
12.	E	C	E	A	D
13.	B	C	C	A	D
14.	B	D	D	E	C
15.	D	A	D	E	B
16.	A	A	B	E	E
17.	C	E	C	E	A
18.	A	C	A	A	E
19.	D	B	C	B	B
20.	C	B	E	C	B
21.	B	B	B	A	C
22.	D	A	C	B	A
23.	E	B	D	B	B
24.	D	E	B	E	B
25.	D	D	A	D	C

2007 solutions

1. **C** $\frac{2007}{9} + \frac{7002}{9} = \frac{9009}{9} = 1001.$

2. **A** If Sam's birthday falls before 9 November, then the fact that she is aged 30 on 8 November means that she was born in 1977. However, if her birthday falls on 9 November or later then her 31st birthday will fall in 2007, which means that she was born in 1976.

3. **B** In general, $(n-1)\times(n+1)-n^2 = n^2-1-n^2 = -1$. This applies with $n = 2007$.

4. **C** $\angle WPQ = 120°$ (interior angle of a regular hexagon),

so $\angle WPS = (360 - 120 - 90)° = 150°$,

Now $PW = PQ$ (sides of a regular hexagon) and $PS = PQ$ (sides of a square) so $PW = PS$. Therefore triangle PSW is isosceles and $\angle PSW = (180-150)° \div 2 = 15°$.

5. **E** $4^4 = (2^2)^4 = 2^8$; $8^{8/3} = (2^3)^{8/3} = 2^8$; $16^2 = (2^4)^2 = 2^8$. However, $32^{6/5} = (2^5)^{6/5} = 2^6$.

6. **A** Let the number of five-pence coins be x. Then $5x + 2(50 - x) = 181$, that is $3x = 81$, that is $x = 27$. So there are 27 five-pence coins and 23 two-pence coins.

7. **D** There are 1003 whole numbers between 1 and 2007 which are divisible by 2. Those which are also divisible by 7 are the multiples of 14, namely 14, 28, 42, …, 2002. There are 143 of these, so the required number is $1003 - 143 = 860$.

8. **E** The distance travelled to Birmingham by the train was 300 km. The time taken to travel this distance at an average speed of 90 km/hr is $\frac{300}{90}$ hr $= 3\frac{1}{3}$ hr $= 3$ hr 20 min. So the train was waiting for 20 minutes.

9. **E** Let the original cost price and original selling price of the dress be C and S respectively. Then $0.8 \times S = 1.04 \times C$. So $S = \frac{1.04}{0.8} \times C = 1.3 \times C$. Therefore the shopkeeper would have made a profit of 30% by selling the dress at its original price.

10. **B** The volume of water which fell at Sprinkling Tarn in 1954 is approximately equal to $(25\,000 \times 6)$ m^3, that is 150 000 m^3. Now $1 \text{ m}^3 = 10^6 \text{ cm}^3 = 10^6 \text{ ml} = 1000$ litres. So approximately 150 million litres of water fell on Sprinkling Tarn in 1954.

11. **D** Each of the original faces of the cube now has area $4 \times 4 - 2 \times 2$, that is 12. In addition, the drilling of the holes has created 24 rectangles, each measuring 2×1. So the required area is $6 \times 12 + 24 \times 2 = 120$.

12. E Let N be the two-digit number 'ab', that is $N = 10a + b$. So the sum of N and its 'reverse' is $10a + b + 10b + a = 11a + 11b = 11(a + b)$. As 11 is prime and a and b are both single digits, $11(a + b)$ is a square if, and only if, $a + b = 11$. So the possible values of N are 29, 38, 47, 56, 65, 74, 83, 92.

13. B The exact number of seconds in six complete weeks is

$$6 \times 7 \times 24 \times 60 \times 60 = 6 \times 7 \times (3 \times 8) \times (2 \times 5 \times 6) \times (3 \times 4 \times 5)$$

$$= 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 10!.$$

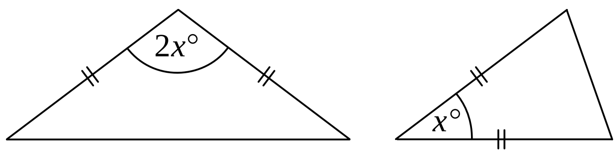
14. B The shaded area is $\frac{x}{360}\left(\pi \times 4^2 - \pi \times 1^2\right) = \frac{15\pi x}{360} = \frac{\pi x}{24}$. So $\frac{\pi x}{24} = \frac{\pi \times 4^2}{6}$; thus $x = 64$.

15. D In the given diagram, there are four hexagons congruent to the hexagon in Figure (i), four hexagons congruent to the hexagon in Figure (ii) and eight hexagons congruent to the hexagon in Figure (iii).

Figure (i) Figure (ii) Figure (iii)

16. A The smallest number of possible prime divisors of 457 that Damien needs to check is the number of prime numbers less than or equal to the square root of 457. Since $21^2 < 457 < 22^2$, he needs to check only primes less than 22. These primes are 2, 3, 5, 7, 11, 13, 17 and 19.

17. C Let the equal sides have length k. The height of the triangle on the left is $k \cos x°$ and its base is $2k \sin x°$, so its area is $k^2 \sin x° \cos x°$. The height of the triangle on the right is $k \sin x°$ and its base is k, so its area is $\frac{1}{2}k^2 \sin x°$. Hence $\cos x° = \frac{1}{2}$ and so $x = 60$.



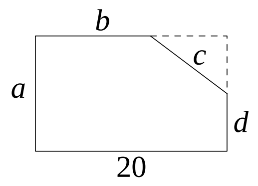
(*Alternatively, the formula* $\Delta = \frac{1}{2}ab \sin C$ *can be used to show that* $\sin x° = \sin 2x°$; *hence* $x + 2x = 180$.)

18. A There are 9 years of the form $123n$ as n may be any digit other than 4. Similarly, there are 9 years each of the forms $234n$, $345n$, $456n$, $567n$ and $678n$, but 10 years of the form $789n$ as, in this case, n may be *any* digit. There are also 9 years of the form $n012$ and 9 of the form $n123$, as in both cases n may be any digit other than 0. However, there are 8 years of the form $n234$ as in this case n cannot be 0 or 1. Similarly, there are 8 years each of the forms $n345$, $n456$, $n567$, $n678$ and $n789$.

So the total numbers of years is $1 \times 10 + 8 \times 9 + 6 \times 8 = 130$.

19. D By the Alternate Segment Theorem $\angle QUS = 55°$. Tangents to a circle from an exterior point are equal, so $QU = QS$ and hence $\angle QSU = \angle QUS = 55°$. So $\angle PQR = 180° - 2 \times 55° = 70°$.

20. C The diagram shows the original rectangle with the corner cut from it to form a pentagon. It may be deduced that the length of the original rectangle is 20 and that a, b, c, d are 8, 10, 13, 15 in some order.



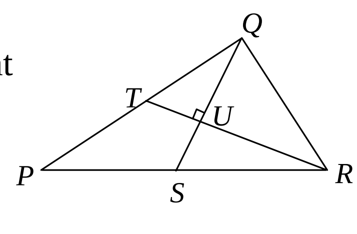
By Pythagoras' Theorem $c^2 = (20 - b)^2 + (a - d)^2$. So c cannot be 8 as there is no right-angled triangle having integer sides and hypotenuse 8. If $c = 10$, then $(20 - b)$ and $(a - d)$ are 6 and 8 in some order, but this is not possible using values of 8, 13 and 15. Similarly, if $c = 15$, then $(20 - b)$ and $(a - d)$ are 9 and 12 in some order, but this is not possible using values of 8, 10 and 13. However, if $c = 13$, then $(20 - b)$ and $(a - d)$ are 5 and 12 in some order, which is true if and only if $a = 15, b = 8, d = 10$.

So the area of the pentagon is $20 \times 15 - \frac{1}{2} \times 5 \times 12 = 270$.

21. B In this solution, the notation p / q / r / s / … represents p beads of one colour, followed by q beads of the other colour, followed by r beads of the first colour, followed by s beads of the second colour etc.

Since the colours alternate, there must be an even number of these sections of beads. If there are just two sections, then the necklace is 4/4 and there is only one such necklace. If there are four, then each colour is split either 2, 2 or 3, 1. So the possibilities are 2/3/2/1 (which can occur in two ways, with the 3 being one colour or the other) or 2/2/2/2 (which can occur in one way) or 3/3/1/1(also one way). Note that 3/2/1/2 appears to be another possibility, but is the same as 2/3/2/1 rotated.
If there are six sections, then each colour must be split into 2, 1, 1 and the possibilities are 2/2/1/1/1/1 (one way)or 2/1/1/2/1/1 (one way). Finally, if there are eight, then the only possible necklace is 1/1/1/1/1/1/1/1. In total that gives 8 necklaces.

22. D Let U be the point of intersection of QS and RT. As QS and RT are medians of the triangle, they intersect at a point which divides each in the ratio 2:1, so $QU = \frac{2}{3} \times 8 = \frac{16}{3}$.

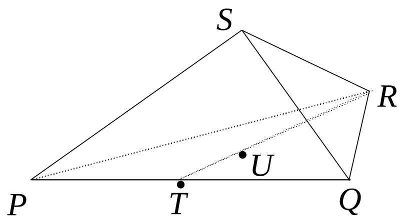


Therefore the area of triangle $QTR = \frac{1}{2} \times RT \times QU = \frac{1}{2} \times 12 \times \frac{16}{3} = 32$.
As a median of a triangle divides it into two triangles of equal area, the area of triangle PTR is equal to the area of triangle QTR, so the area of triangle PQR is 64.

23. E Let the lengths of the sides of the cuboid, in cm, be a, b and c. So $4(a + b + c) = x$. Also, by Pythagoras' Theorem $a^2 + b^2 + c^2 = y^2$. Now the total surface area of the cuboid is

$$2bc + 2ca = (a + b + c)^2 - \left(a^2 + b^2 + c^2\right) = \left(\frac{x}{4}\right)^2 - y^2 = \frac{x^2 - 16y^2}{16}.$$

24. D The diameter of the sphere is $l - 2h$ where l is the length of a space diagonal of the cube and h is the perpendicular height of one of the tetrahedral corners when its base is an equilateral triangle.



The diagram shows such a tetrahedron: S is a corner of the cube; the base of the tetrahedron, which is considered to lie in a horizontal plane, is an equilateral triangle, PQR, of side $\sqrt{2}$ units; T is the midpoint of PQ. Also U is the centroid of triangle PQR, so $RU : UT = 2 : 1$. As U is vertically below S, the perpendicular height of the tetrahedron is SU.

As RTP is a right angle, $RT^2 = RP^2 - TP^2 = (\sqrt{2})^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{3}{2}$. Also, $RU = \frac{2}{3}RT$, so $RU^2 = \frac{4}{9}RT^2 = \frac{4}{9} \times \frac{3}{2} = \frac{2}{3}$.
So $SU^2 = SR^2 - RU^2 = 1 - \frac{2}{3} = \frac{1}{3}$. Therefore $h = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3}$.
Now $l^2 = 2^2 + 2^2 + 2^2 = 12$, so $l = \sqrt{12} = 2\sqrt{3}$. Therefore the diameter of the sphere is $2\sqrt{3} - 2 \times \dfrac{\sqrt{3}}{3} = \dfrac{4\sqrt{3}}{3}$.

25. D As the line $y = x$ is an axis of symmetry of the curve, if the point (a, b) lies on the curve, so too does the point (b, a). Hence the equation of the curve may also be written as $x = \dfrac{py + q}{ry + s}$.

Therefore, substituting for x in the original equation:

$$y = \frac{p\left(\dfrac{py + q}{ry + s}\right) + q}{r\left(\dfrac{py + q}{ry + s}\right) + s} = \frac{p(py + q) + q(ry + s)}{r(py + q) + s(ry + s)}.$$

Therefore $$y\big(r(py + q) + s(ry + s)\big) = p(py + q) + q(ry + s),$$

that is $$y^2 r(p + s) + y\left(qr + s^2 - p^2 - qr\right) - pq - qs = 0,$$

that is $$(p + s)\left(y^2 r + y(s - p) - q\right) = 0.$$

Since r is non-zero, the expression in the second bracket is non-zero for all but at most two values of y. Hence $p + s = 0$.

2008 solutions

1. **C** $2 \times 2008 + 2008 \times 8 = 10 \times 2008 = 20080$.

2. **B** The cost per pound is £$\frac{255}{1250} \approx$ £$\frac{1}{5} = 20$ p.

3. **D** $\frac{1}{2^6} + \frac{1}{6^2} = \frac{3^2 + 2^4}{2^6 \times 3^2} = \frac{25}{2^6 \times 3^2} = \frac{5^2}{(2^3 \times 3)^2}$. Hence the answer is $\frac{5}{2^3 \times 3} = \frac{5}{24}$.

4. **C** From the units column we see that $S = 0$. Then the tens column shows that $R = 9$, the hundreds column that $Q = 1$, and the thousands that $P = 6$. So $P + Q + R + S = 16$.

5. **E** Since 1% of £400 = £4, the total VAT charged was £4 $\times$ 17.5 = £70, giving a total cost of £400 + £70 = £470. Therefore the minimum number of entries needed is 94.

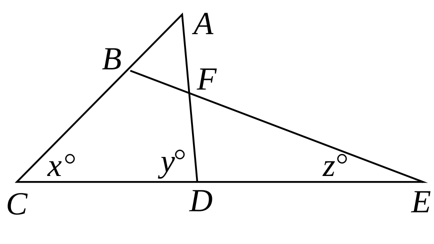
6. **E**

6		
4	5	
1	2	3

We number the squares to identify them. The only line of symmetry possible is the diagonal through 1 and 5. For a symmetric shading, if 4 is shaded, then so too must be 2; so either both are shaded or neither. Likewise 3 and 6 go together and provide 2 more choices. Whether 1 is shaded or not will not affect a symmetry, and this gives a further 2 choices; and the same applies to 5. Overall, therefore, there are $2^4 = 16$ choices. However, one of these is the choice to shade no squares, which is excluded by the question.

7. **D** In 1.8 miles there are 1.8×5280 feet $= 18 \times 528$ feet, while in 8 months there are roughly $8 \times 30 \times 24 \times 60$ minutes. Hence the time to 'run' one foot in minutes is roughly $\frac{10 \times 30 \times 20 \times 60}{20 \times 500} = 36$ minutes.

8. **A** In triangle ACD, $\angle CAD = (180 - x - y)°$.
As $AB = AF$, triangle ABF is isosceles hence $\angle ABF = \angle AFB = \frac{1}{2}(x + y)°$.
Thus $\angle DFE = \angle AFB = \frac{1}{2}(x + y)°$ (vertically opposite angles). Now in triangle DFE, $\angle FDE = (180 - y)°$. Hence $z° = 180° - \angle DFE - \angle FDE = \frac{1}{2}(y - x)°$.



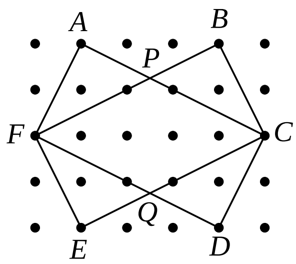
9. **D** A number is divisible by 9 if, and only if, the sum of its digits is divisible by 9. The given number is $N + 2$, where $N = 222\ldots 220$ has 2007 2s. Since $2007 = 223 \times 9$, N is divisible by 9 and the required remainder is therefore 2.

10. D By inspection

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{4}; \qquad \frac{3}{5} = \frac{1}{2} + \frac{1}{10}; \qquad \frac{3}{6} = \frac{1}{3} + \frac{1}{6}; \qquad \frac{3}{8} = \frac{1}{4} + \frac{1}{8}.$$

However $\frac{3}{7} \neq \frac{1}{m} + \frac{1}{n}$. [To see why, suppose that $\frac{3}{7} = \frac{1}{m} + \frac{1}{n}$ and note that $\frac{1}{m} > \frac{1}{n}$ or vice versa. We will suppose the former. Then $\frac{1}{m} \geqslant \frac{3}{14} > \frac{3}{15}$ and so $\frac{1}{m} > \frac{1}{5}$ and $m < 5$. Also $\frac{1}{m} < \frac{3}{7}$ and so $3m > 7$. Hence $m \geqslant 3$. So $m = 4$ or $m = 3$. However $\frac{3}{7} - \frac{1}{4} = \frac{5}{28}$ and $\frac{3}{7} - \frac{1}{3} = \frac{2}{21}$ neither of which has the form $\frac{1}{n}$.]

11. B Let the six points where lines meet on the dot lattice be A, B, C, D, E, F as shown and let the other two points of intersection be P (where AC and BF meet) and Q (where CE and DF meet).

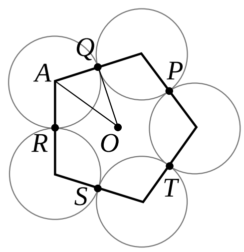


Triangles APB and CPF are similar with base lengths in the ratio 3:5. Hence triangle CPF has height $\frac{5}{8} \times 2 = \frac{5}{4}$ units and base length 5 units so that its area is $\frac{1}{2} \times \frac{5}{4} \times 5$ square units. Since the same is true of triangle CQF, the required area is $\frac{5}{4} \times 5 = 6\frac{1}{4}$ square units.

12. C There are 365 days in a normal year and 366 in a leap year. Apart from certain exceptions (none of which occurs in this period) a leap year occurs every 4 years. Now $365 = 7 \times 52 + 1$ and $366 = 7 \times 52 + 2$. Hence each date moves on by 5 days every 4 years. So in 60 years, it moves on 75 days. Since $75 = 7 \times 10 + 5$, that means it moves on to a Thursday.

13. C Since $1280 = 2^8 \times 5 = 2^8(2^0 + 2^2) = 2^8 + 2^{10}$, we may take m=8 and $n = 10$ (or vice versa) to get $m + n = 8 + 10 = 18$. It is easy to check that there are no other possibilities.

14. D The internal angle of a regular pentagon is 108°. Let A be the centre of a touching circle, as shown. Since OA bisects $\angle RAQ$, $\angle OAQ = 54°$. Also, triangle OAQ is right-angled at Q (radius perpendicular to tangent). Since $AQ = 1$, $OQ = \tan 54°$.



15. A The sequence proceeds as follows: 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1 … . The block 4, 2, 1 repeats *ad infinitum* starting after t_7. But $2008 - 7 = 2001$ and $2001 = 3 \times 667$. Hence t_{2008} is the third term in the 667th such block and is therefore 1.

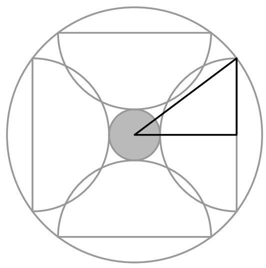
16. A Adding the three given equations gives $4(x + y + z) = 3000$. Therefore $x + y + z = 750$. So the mean is $\frac{750}{3} = 250$.

17. E Let 'X' be a single digit. If $2008 - 200X = 2 + 0 + 0 + X$ then $8 - X = 2 + X$ so $X = 3$. So Alice (being the younger) could have been born in 2003. Next if $2008 - 199X = 1 + 9 + 9 + X$ then $18 - X = 19 + X$, which is impossible. Similarly if $2008 - 198X = 1 + 9 + 8 + X$ then $28 - X = 18 + X$, so $X = 5$. Thus Alice or Andy could have been born in 1985. Finally if $2008 - 19YX = 1 + 9 + X + Y$ for some digit $Y \leqslant 7$, then $108 - YX = 10 + Y + X$. Hence $98 = YX + Y + X$ which is impossible, since $YX + Y + X$ is at most $79 + 7 + 9 = 95$. Hence there are no more possible dates and so Andy was born in 1985 and Alice in 2003.

18. C Since $XY^2 = 18$, $YZ^2 = 32$ and $XZ^2 = 50$, we have $XZ^2 = XY^2 + YZ^2$. Hence by the converse of Pythagoras' Theorem, $\angle XYZ = 90°$. Since the angle in a semi-circle is 90° the segment XZ is the diameter of the specified circle. Hence the radius is $\frac{1}{2}\sqrt{50}$ and the area of the circle is $\dfrac{50\pi}{4} = \dfrac{25\pi}{2}$.

19. B Let $199p + 1 = X^2$. Then $199p = X^2 - 1 = (X + 1)(X - 1)$. Note that 197 is prime. If p is also to be prime then **either** $X + 1 = 199$, in which case $X - 1 = 197$, **or** $X - 1 = 199$, in which case $X + 1 = 201$ (and $201 = 3 \times 67$ is not prime). Note that $X - 1 = 1$, $X + 1 = 199p$ is impossible. Hence $p = 197$ is the only possibility.

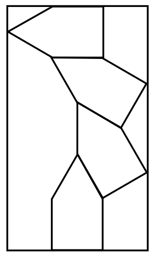
20. B Let r_1, r_2 and r_3 be the radii of the shaded circle, semicircles and outer circle respectively. A right-angled triangle can be formed with sides r_3, $(r_1 + r_2)$ and r_2.



Hence, by Pythagoras' Theorem, $r_3^2 = (r_1 + r_2)^2 + r_2^2$.
Now $\pi r_1^2 = 4$, hence $r_1 = 2/\sqrt{\pi}$. Likewise $r_2 = 6/\sqrt{\pi}$.
Hence $r_2 = 3r_1$ so that $r_3^2 = (r_1 + 3r_1)^2 + (3r_1)^2 = 25r_1^2$.
Thus the required area is $25 \times 4 = 100$.

21. B Since 2008/1998 lies between 1 and 2, $a = 1$. Subtracting 1 and inverting gives $b + 1/(c + 1/d) = 1998/10 = 199 + 4/5$ so that $b = 199$. Then $1/(c + 1/d) = 4/5$ so that $c + 1/d = 5/4$ and this gives $c = 1$ and $d = 4$. {*Note : This is an example of a 'continued fraction'.*}

22. A Let r be the length of a side of the equilateral triangle.



Hence the width of the rectangle is $r\sin 60° + r + r\sin 60° = r(1 + 2\sin 60°) = r(1 + \sqrt{3})$ and its length is $3r + 2r\sin 60° = r(3 + \sqrt{3})$.
So the ratio of the length to the width is

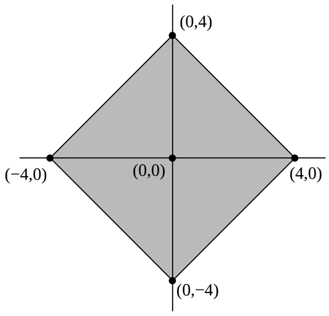
$$(3 + \sqrt{3}) : (1 + \sqrt{3}) = \sqrt{3}(1 + \sqrt{3}) : (1 + \sqrt{3}) = \sqrt{3} : 1.$$

23. B Let $X = x + 3$ and $Y = y - 3$. Then the given equation becomes $(X + Y)^2 = XY$.
So $X^2 + XY + Y^2 = 0$. However X^2, Y^2 and $XY\ (= (X + Y)^2)$ are non-negative. Hence $X = Y = 0$; so $x = -3$ and $y = 3$ is the only solution.

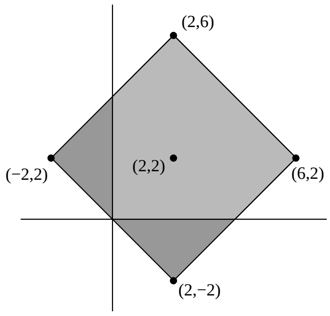
24. **E** $1 + 3 + 5 + 7 + \ldots + (2n + 1) = (n + 1)^2$. The n in the three cases given is 12, $\frac{1}{2}(x - 1)$ and $\frac{1}{2}(y - 1)$. So, the triangle has sides of length $12 + 1$, $\frac{1}{2}(x - 1) + 1$ and $\frac{1}{2}(y - 1) + 1$. However the only right-angled triangle having sides of whole number length with hypotenuse 13 is the (5, 12, 13) triangle. So $x = 9$ and $y = 23$ (or vice versa). Hence $x + y = 32$.

25. **D** To work out the area of $\left||x| - 2\right| + \left||y| - 2\right| \leqslant 4$, we first consider the region $|x| + |y| \leqslant 4$ which is shown in (a). This region is then translated to give $|x - 2| + |y - 2| \leqslant 4$ as shown in (b).

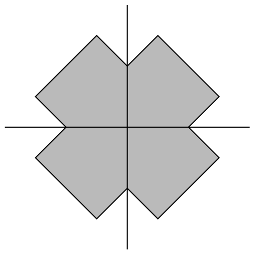
By properties of the modulus, if the point (x, y) lies in the polygon, then so do $(x, -y)$, $(-x, y)$ and $(-x, -y)$. Thus $\left||x| - 2\right| + \left||y| - 2\right| \leqslant 4$ can be obtained from (b) by reflecting in the axes and the origin, as shown in (c).



(a)



(b)



(c)

Hence the required area is 4 times the area in the first quadrant. From (b), the required area in the first quadrant is the area of a square of side $4\sqrt{2}$ minus two triangles (cut off by the axes) which, combined, make up a square of side $2\sqrt{2}$. So the area in the first quadrant is

$(4\sqrt{2})^2 - (2\sqrt{2})^2 = 32 - 8 = 24$.

Hence the area of the polygon is $4 \times 24 = 96$ square units.

2009 solutions

1. **A** 20% of 30% = $0.2 \times 0.3 = 0.06 = 6\%$.

2. **D** $\frac{785}{15} = 52\frac{1}{3}$ hence 785 is not a multiple of 15. But $\frac{135}{15} = 9$, $\frac{315}{15} = 21$, $\frac{555}{15} = 37$, $\frac{915}{15} = 61$.

3. **E** $1 - 32 + 81 - 64 + 25 - 6 = 5$.

4. **E** Steve achieved $\frac{150}{10} \times 4.5$ miles per gallon which is $15 \times 4.5 = 67.5 \approx 70$.

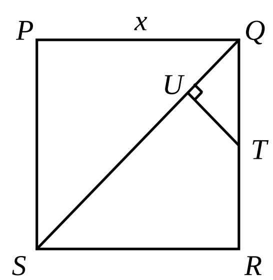
5. **A** As the ratio of the radii is 3 : 4 then the number of revolutions made by the larger wheel is $120000 \times \frac{3}{4} = 90000$.

6. **C** If at most two marbles of each colour are chosen, the maximum number we can choose is 8, corresponding to 2 of each. Therefore, if 9 are chosen, we must have at least 3 of one colour, but this statement is not true if 9 is replaced by any number less than 9.

7. **B** The top left 2 by 2 outlined block must contain a 3 and a 4 and this can be done in two ways. For each choice there is only one way to complete the entire mini-sudoku.

8. **C** The increase in entries from 2007 to 2008 is $92\,690 - 87\,400 = 5290$.
Hence the percentage increase is
$\frac{5290}{87400} \times 100\% = \frac{5290}{874}\% \approx \frac{5400}{900}\% = 6\%$.
(The exact value is $6\frac{1}{19}$.)

9. **D** As T is the midpoint of QR then $QT = \frac{1}{2}x$.
Since $\angle UQT = \angle SQR = 45°$ and $\angle QUT = 90°$, $\angle UTQ = 45°$. Thus triangle QTU is isosceles with $UQ = UT$.
In triangle QTU, by Pythagoras' Theorem, $QT^2 = QU^2 + TU^2$.
Hence $\left(\frac{1}{2}x\right)^2 = 2TU^2$ so $TU^2 = \frac{1}{8}x^2$ giving $TU = \frac{x}{2\sqrt{2}}$.



10. **B** A number is a multiple of 6 precisely when it is both a multiple of 2 and of 3. To be a multiple of 2, it will need to end with an even digit; i.e. 0 or 2. If it ends with 0, the sum of the other two digits must be a multiple of 3; and only $3 = 1 + 2$ or $6 = 1 + 5$ are possible. That gives the numbers 120, 210, 150, 510. If it ends with 2, the sum of the others must be $1 = 0 + 1$ or $4 = 1 + 3$. That gives 102, 132 and 312.

11. C $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2} = 2^2 \times 2^{1/2} = 2^{2\frac{1}{2}}$. Hence $x = 2\frac{1}{2}$.

12. E $\cos 50° < \sin 50° < 1$. Hence

$$\frac{1}{\cos 50°} > \frac{1}{\sin 50°} > 1 > \sin 50° > \cos 50°.$$

$\tan 50° = \dfrac{\sin 50°}{\cos 50°} < \dfrac{1}{\cos 50°}$ hence $\dfrac{1}{\cos 50°}$ has the greatest value.

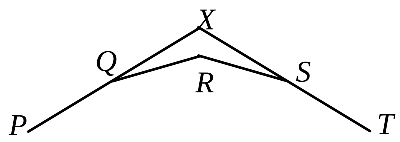
13. C $x - \dfrac{1}{x} = y - \dfrac{1}{y}$ hence $x^2y - y = xy^2 - x$. Thus

$xy(y - x) + y - x = 0$.

Therefore $(y - x)(xy + 1) = 0$. As $x \neq y$ then $y - x \neq 0$.

Hence $xy + 1 = 0$ giving $xy = -1$.

14. D Let the external angle of the regular polygon be $x°$.



Hence $\angle XQR = \angle XSR = x°$ and reflex angle $\angle QRS = (180 + x)°$.

As the sum of the angles in the quadrilateral $QRSX$ is

$360°$ then $140 + x + x + 180 + x = 360$.

Hence $3x = 40$ and the polygon has $\dfrac{360}{40 \div 3} = 27$ sides.

15. D Let $\dfrac{n}{100 - n} = x$ where x is an integer. Hence $n = 100x - nx$.

Hence $n(1 + x) = 100x$ giving $n = \dfrac{100x}{1 + x}$.

Now x and $1 + x$ can have no common factors. Therefore $1 + x$ must be a factor of 100 and can be any of them.

Hence $1 + x \in \{\pm 1, \pm 2, \pm 4, \pm 5, \pm 10, \pm 20, \pm 25, \pm 50, \pm 100\}$ thus the number of possible integers n is 18.

16. B Since $x^4 - y^4 = 2009$ it follows that $(x^2 + y^2)(x^2 - y^2) = 2009$.

But $x^2 + y^2 = 49$ hence $x^2 - y^2 = \dfrac{2009}{49} = 41$.

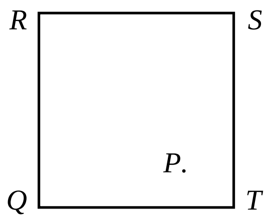
Subtracting gives $2y^2 = 8$ hence $y^2 = 4$. Since $y > 0$, $y = 2$.

17. C The greatest possible value of f is achieved by a rectangular cut through an edge of a cube and the furthest edge from it. If we take x as the side of the cube, by Pythagoras' Theorem the extra surface area formed by the cut is $2\sqrt{2}x^2$. Hence $f = \dfrac{2\sqrt{2}x^2}{6x^2} = \dfrac{\sqrt{2}}{3}$.

18. A We have $y^2 = x(2 - x)$. Now $y^2 \geqslant 0$ for all real y hence $x(2 - x) \geqslant 0$. Hence $0 \leqslant x \leqslant 2$. In fact we can rewrite the equation as $(x - 1)^2 + y^2 = 1$; so this is a circle of radius 1 with centre (1,0).

19. **C** The distance cycled by Hamish between noon and 4 pm is $4x$.
The distance cycled by Ben between 2 pm and 4 pm is $2y$.
They meet at 4 pm hence $4x + 2y = 51$ or $2x + 2(x + y) = 51$ (*).
If they had both started at noon then they would have met at 2:50 pm and so $2\frac{5}{6}(x + y) = 51$.
Hence $x + y = 51 \times \frac{6}{17} = 18$. Hence from (*) $2x + 2 \times 18 = 51$.
Hence $2x = 15$ giving $x = 7\frac{1}{2}$. Thus $y = 10\frac{1}{2}$.

20. **E** If $\angle RPQ = 90°$ then P lies on a semicircle of diameter RQ.



Let x be the side-length of the square $QRST$.
Hence the area of the semicircle $RPQ = \frac{1}{2}\pi\left(\frac{1}{2}x\right)^2 = \frac{1}{8}\pi x^2$ and the area of square $QRST$ is x^2.
$\angle RPQ$ is acute when P is outside the semicircle RPQ.

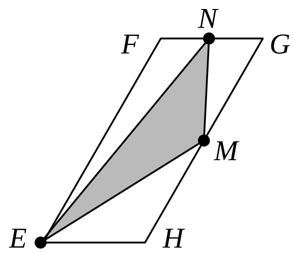
Hence the probability that $\angle RPQ$ is acute is $\dfrac{x^2 - \frac{1}{8}\pi x^2}{x^2} = 1 - \dfrac{\pi}{8}$.

21. **B** Let r be the radius of the small cone and h the height.
Let l_1 and l_2 be the slant heights of the small and large cones respectively.
By Pythagoras' Theorem $l_2 = \sqrt{6^2 + 8^2} = 10$.
Using similar triangles, $\dfrac{l_1}{r} = \dfrac{10}{6}$ so $l_1 = \dfrac{5}{3}r$ and $\dfrac{h}{8} = \dfrac{r}{6}$ giving $h = \dfrac{4}{3}r$.
Thus the area of the curved surface of the frustum is

$$\pi \times 6 \times 10 - \pi \times r \times \frac{5}{3} \times r = \pi\left(60 - \frac{5r^2}{3}\right).$$

The sum of the areas of the two circles is $\pi \times 6^2 + \pi \times r^2 = \pi(36 + r^2)$.
Hence $\pi\left(60 - \dfrac{5r^2}{3}\right) = \pi(36 + r^2)$ and so $24 = \dfrac{8r^2}{3}$ giving $r = 3$, so $h = \dfrac{4}{3} \times 3 = 4$. Therefore, in cms, the height of the frustum is $8 - 4 = 4$.

22. **C** Let the perpendicular distance between EH and FG be x cm and the area of the parallelogram $EFGH$ be y cm^2. Thus $y = FG \times x$.



The area of triangle EFN is $\frac{1}{2}FN \times x = \frac{1}{2} \times \frac{1}{2} \times FG \times x = \frac{1}{4}y$ cm^2.
Likewise the areas of triangles EHM and NGM are $\frac{1}{4}y$ cm^2 and $\frac{1}{8}y$ cm^2 respectively.
The area of triangle ENM is 12 cm^2, hence $y = 12 + \frac{5}{8}y$ and so $y = 32$. Hence the area of the parallelogram $EFGH$ is 32 cm^2.

23. **D** Label the rows of the triangles from left to right as follows: $a_1, \ldots, a_5$; $b_1, \ldots, b_{10}$ and $c_1, \ldots, c_5$.

Now 1 cannot be at a_4, a_5, b_7, b_8 or c_4 hence 1 must be at c_3.

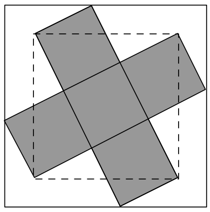
Hence b_4 and b_5 are 2 and 5 in either order. Hence a_3 is 1 or 4.

But 1 cannot be at a_4 or b_7 hence 1 must be at a_3.

4 cannot be at b_3 thus 4 is at a_2.

Hence the number on the face with the question mark must be 4.

24. **B** A shaded triangle is congruent to an unshaded triangle (ASA).



Hence the area of the dashed square is equal to the area of the cross and both are 5.

Thus the side-length of the dashed square is $\sqrt{5}$.

Hence the sides of a shaded triangle are: $\frac{1}{2}$, 1 and $\frac{1}{2}\sqrt{5}$.

Now the perpendicular distance between the squares is equal to the altitude, h, of the shaded triangle. The area of such a triangle is $\frac{1}{2} \times \left(\frac{1}{2} \times 1\right) = \frac{1}{4}$ so that $\frac{1}{2} \times \left(\frac{1}{2}\sqrt{5} \times h\right) = \frac{1}{4}$ which gives $h = \dfrac{1}{\sqrt{5}}$.

Hence the length of the sides of the outer square are
$\sqrt{5} + 2 \times \dfrac{1}{\sqrt{5}} = \dfrac{5}{\sqrt{5}} + \dfrac{2}{\sqrt{5}} = \dfrac{7}{\sqrt{5}}$.

Thus the area of the large square is $\left(\dfrac{7}{\sqrt{5}}\right)^2 = \dfrac{49}{5}$.

25. **A** The left-hand side of the equation can be written as

$$(a + 1)(b + 1)(c + 1)(d + 1) - 1.$$

Hence

$$(a + 1)(b + 1)(c + 1)(d + 1) = 2010.$$

Now expressing 2010 as a product of primes gives
$2010 = 2 \times 3 \times 5 \times 67$ hence
$a + b + c + d = 1 + 2 + 4 + 66 = 73$.

2010 solutions

1. **C** The only two-digit cubes are 27 and 64. As 1 *Down* is one less than a cube then 3 *Across* must start with 6 or 3 and so is 64. Thus $x = 4$.

2. **B** The smallest possible value is attained by using $p = 1$, $q = 2$ and $r = 3$. Therefore this value is $20 \times 1 + 10 \times 2 + 3 = 43$.

3. **C** The three internal angles of an equilateral triangle are all 60°. As the sum of the angles on a straight line is 180° then the sum of the four marked angles is $2 \times (180 - 60)° = 2 \times 120° = 240°$.

4. **B** $2 + 0 + 1 + 1 = 4$. Multiples of 4 are even, hence 2011 is not valid and the same argument applies to 2013, 2015, 2017 and 2019.

$2 + 0 + 1 + 2 = 5$. The units digit for multiples of 5 is 0 or 5, hence 2012 is not valid.

$2 + 0 + 1 + 4 = 7$. But $\frac{2014}{7} = 287\frac{5}{7}$, hence 2014 is not valid.

$2 + 0 + 1 + 6 = 9$. Since $2016 = 9 \times 224$, 2016 is valid.

Hence we have to wait $2016 - 2010$ (= 6) more years.

5. **D** If the statement is true then the capacity (in litres) of Morecambe Bay is approximately:

$$20 \times 10^6 \times 365 \times 24 \times 60 \times 6 = 10^8 \times (6 \times 365) \times (2 \times 24) \times 6$$
$$\approx 6 \times 10^8 \times 2000 \times 50 = 6 \times 10^{13}.$$

6. **B** The length of the road is 8 km. Hence the time taken to run down the mountain is $\frac{8}{12}$ hours = $\frac{8}{12} \times 60$ min = 40 min.

7. **C** There are 24 arrangements of the letters in the word ANGLE with A as the first letter. In alphabetical order AEGLN is first and ANLGE is last ie 24th. ANLEG is the 23rd and hence ANGLE is the 22nd.

8. **D** $(x + y + z)(x - y - z) = [x + (y + z)][x - (y + z)] = x^2 - (y + z)^2$.

9. **D** $(2 \diamond 3) \diamond 4 = (2^3 - 3^2) \diamond 4 = (-1) \diamond 4 = (-1)^4 - 4^{-1} = 1 - \frac{1}{4} = \frac{3}{4}$.

10. **E** Let the original square have sides of length y cm and the single square which is not 1×1 have sides of length x cm. Then $y^2 = 36 + x^2$, and so $y^2 - x^2 = 36$ and hence $(y + x)(y - x) = 36$.

As $36 = 2^2 \times 3^2$ and $y + x > y - x$ the possible factors of 36 are:

$y + x$	$y - x$	y	x	
9	4	$6\frac{1}{2}$	$2\frac{1}{2}$	impossible
12	3	$7\frac{1}{2}$	$4\frac{1}{2}$	impossible
18	2	10	8	possible
36	1	$18\frac{1}{2}$	$17\frac{1}{2}$	impossible

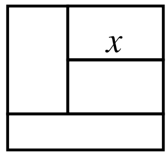
We can check that $10^2 = 36 + 8^2 = 100$ and hence the length of the side of the *original* square is 10 cm.

11. D Squaring the numbers given allows us to see their order easily:

$(9\sqrt{2})^2 = 81 \times 2 = 162$	$(3\sqrt{19})^2 = 9 \times 19 = 171$	$(4\sqrt{11})^2 = 16 \times 11 = 176$
$(5\sqrt{7})^2 = 25 \times 7 = 175$	$(6\sqrt{5})^2 = 36 \times 5 = 180$	

As 175 is the middle one of these numbers, the answer is $5\sqrt{7}$.

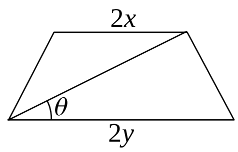
12. A As the square has side length 1 its area is $1 \times 1 = 1$.



Thus the area of each of the four rectangles is $\frac{1}{4}$.
The length of the bottom rectangle is 1 hence its width is $\frac{1}{4}$.
Thus the width of each of the two congruent rectangles is $\frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{8}$.
Hence the area of one of these congruent rectangles is $\frac{3}{8}x$.
But we know this area is $\frac{1}{4}$, therefore $\frac{3}{8}x = \frac{1}{4}$ and hence $x = \frac{2}{3}$.

13. A The lowest common multiple of 3 and 4 is 12. Hence both of the required conditions are satisfied only by numbers that are 2 less than multiples of 12 and also less than 100, ie: 10, 22, 34, 46, 58, 70, 82 and 94.
Therefore 8 two-digit numbers satisfy the conditions.

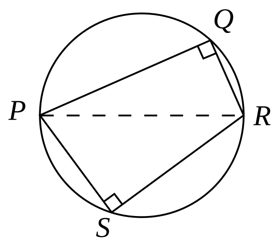
14. E Drop perpendiculars from the top vertices to the bottom line. The distance from the foot to the nearer base vertex is $\frac{1}{2}(2y - 2x) = y - x$. So the distance to the further base vertex is $2y - (y - x) = y + x$.



Hence $\cos\theta = \dfrac{x + y}{d}$ where d is the length of the diagonal.

15. E The first eight prime numbers are: 2, 3, 5, 7, 11, 13, 17, 19.
If the sum of two prime numbers is prime, one of them must be 2.
If the sum of three different prime numbers is prime they must all be odd.
The answer is therefore 19 as: $2 + 17 = 19$ and $3 + 5 + 11 = 19$.

16. E As PR is a diameter, $\angle PQR = \angle PSR = 90°$ (angles in a semicircle are 90°).



Since $PQ = 12 \times 5$ and $QR = 5 \times 5$, triangle PQR is an enlarged 5, 12, 13 triangle and so $PR = 13 \times 5 = 65$.
Since $PR = 5 \times 13$ and $SR = 4 \times 13$, triangle PRS is an enlarged 3, 4, 5 triangle and so $SP = 3 \times 13 = 39$.

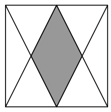
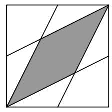
17. E $\sqrt{9^{16x^2}} = 9^{(16x^2)/2} = 9^{8x^2}$.

18. A Let x be the length of the shaded rectangle.

By Pythagoras' Theorem, $x^2 = 2^2 + 2^2$, hence $x = 2\sqrt{2}$.

The total surface area of the two prisms equals the surface area of the solid cube plus twice the surface area of that shaded rectangle, that is $6 \times 2 \times 2 + 2 \times 2 \times 2\sqrt{2} = 24 + 8\sqrt{2} = 8(3 + \sqrt{2})$.

19. B In the rhombus on the left, drawing vertical straight lines at distances of $1\frac{1}{2}$, 3 and $4\frac{1}{2}$ from the left edge of the square, and a horizontal straight line bisecting the square, creates 16 equivalent triangles. Of these, four are shaded giving a total shaded area of $\frac{1}{4} \times 6 \times 6 = 9$.

Draw in the diagonal from NW to SE in the rhombus on the right. The four unshaded triangles now above the shaded area are all equal in area (a say); and one can see that 3 of these together make up $\frac{1}{4}$ of the square. Hence $a = 3$. Thus the shaded area equals $36 - 3 \times 8 = 12$.

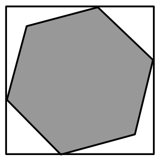
Therefore the difference between the shaded areas is $12 - 9 = 3$.

20. C Let the number of boys in the class be x. Hence $\dfrac{10}{10 + x} \times \dfrac{9}{9 + x} = \dfrac{3}{20}$.

Simplifying gives $1800 = 3(10 + x)(9 + x)$ and then $x^2 + 19x - 510 = 0$.

Factorising gives $(x + 34)(x - 15) = 0$ and, since $x \neq -34$, $x = 15$.

21. A The hypotenuse of one of the small right-angled triangles is parallel to the diagonal and hence makes angles of 45°. Since the hypotenuse has length 1, the other two sides have length $\frac{1}{\sqrt{2}}$, by Pythagoras' Theorem. As the internal angle of a regular hexagon is 120°, drawing a diagonal from NW to SE forms two triangles, bottom right, each with angles 45°, 120° and 15°. (The sum of the angles in a triangle is 180°).



Let the square have length y units. Using the Sine Rule gives

$$\frac{y - \frac{1}{\sqrt{2}}}{\sin 120°} = \frac{1}{\sin 45°}.$$

Hence $y - \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{2} \div \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{3}}{\sqrt{2}}$ and therefore $y = \dfrac{\sqrt{3} + 1}{\sqrt{2}}$.

Hence the area of the square is $y^2 = \left(\dfrac{\sqrt{3} + 1}{\sqrt{2}}\right)^2 = \dfrac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$.

22. B Since $x^2 - px - q = 0$, it follows that $x^3 = px^2 + qx$.

But $x^2 = px + q$ and so $x^3 = p(px + q) + qx$, ie $x^3 = (p^2 + q)x + pq$.

The three possible values shown for pq are 3, 5 and 7.

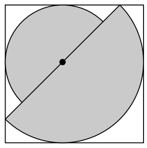
If $pq = 3$, $p^2 + q = 1^2 + 3 = 4$ or $p^2 + q = 3^2 + 1 = 10$. Hence $4x + 3$ and $10x + 3$ could equal x^3.

If $pq = 7$, we may take $p = 1$, $q = 7$ to get $p^2 + q = 1^2 + 7 = 8$. Hence $8x + 7$ could equal x^3.

If $pq = 5$, we may take $p = 5$, $q = 1$ to get $p^2 + q = 5^2 + 1 = 26$. Hence $26x + 5$ could equal x^3.

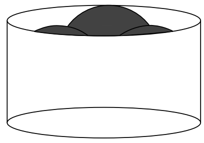
However, the only other possibility, $p = 1$, $q = 5$ gives $p^2 + q = 6 \neq 8$. Therefore $8x + 5 \neq x^3$.

23. B Let r_1 and r_2 represent the radii of the smaller and larger semicircles respectively. A vertical line through the common centre of the semicircles gives $r_1 + r_2 = 2$ … (1). Also, together with the diameter of the larger semicircle, this line forms a right-angled, isosceles triangle giving $\sin 45° = \frac{r_1}{r_2}$.



Hence $r_2 = \sqrt{2}r_1$ … (2).
Substituting (2) into (1) gives $(1 + \sqrt{2})\,r_1 = 2$ so that $r_1 = 2(\sqrt{2} - 1)$.
Therefore $r_2 = 2\sqrt{2}(\sqrt{2} - 1)$.
Hence the total shaded area is
$\frac{1}{2}\pi(r_1^2 + r_2^2) = \frac{1}{2}\pi\left[4(\sqrt{2} - 1)^2 + 8(\sqrt{2} - 1)^2\right] = 6\pi(3 - 2\sqrt{2})$.

24. E The volume of the three spheres is $3 \times \frac{4}{3}\pi \times 1^3 = 4\pi$.
Let r be the radius of the cross-sectional area of the cylinder.
Hence the volume of the cylinder is $2\pi r^2$.
Thus the required fraction is $\frac{2}{r^2}$.
The straight lines joining the centres of the three spheres form an equilateral triangle of side length 2.



Let x be the distance from the centre of a sphere to the midpoint of the triangle. Using the Sine Rule, $\frac{2}{\sin 120°} = \frac{x}{\sin 30°}$ hence $x = \frac{2}{\sqrt{3}}$.
As the sphere has radius 1, $r = x + 1$ and $r = 1 + \frac{2}{\sqrt{3}}$.
Thus $r^2 = \frac{1}{3}(2 + \sqrt{3})^2 = \frac{1}{3}(7 + 4\sqrt{3})$. Hence the required fraction is $\frac{6}{7 + 4\sqrt{3}}$.

25. D The sum of 10 different digits is 45. As the sum of the digits in the question is 36 then digits adding to 9 are omitted.
The combinations of digits satisfying this are:

$$9;\ 1 + 8;\ 2 + 7;\ 3 + 6;\ 4 + 5;\ 1 + 2 + 6;\ 1 + 3 + 5;\ 2 + 3 + 4.$$

When '0' is not involved there are $(8! + 4 \times 7! + 3 \times 6!)$ numbers, whereas when '0' is used there are $(8 \times 8! + 4 \times 7 \times 7! + 3 \times 6 \times 6!)$.
This gives a total of
$9 \times 8! + (4 + 28) \times 7! + (3 + 18) \times 6! = (72 + 32 + 3) \times 7! = 107 \times 7!$
Hence $N = 107$.

2011 solutions

1. **D** Every integer is divisible by 1; 2012 is divisible by 2 since it is even; 2013 is divisible by 3 since its digits total to a multiple of 3; and 2015 is divisible by 5 since its last digit is 5. However, 2014 is not divisible by 4 because 14 is not.

2. **D** After the first spill, $\frac{1}{3}$ of the water remains.

After the second spill, $\frac{3}{5} \times \frac{1}{3}$ of the water remains, hence $\frac{1}{5}$ of the pail had water left in it.

3. **B** After 62 stages of the process *Lumber9* has taken $1 + 2 + 3 + \ldots + 62 = 1953$ steps and has reached the number $1 - 2 + 3 - \ldots - 62 = -31$. After taking another 58 steps in the positive direction, it has taken $1953 + 58 = 2011$ steps in total, and has reached the number $-31 + 58 = 27$.

4. **D** Since $3^1 = 3$, $3^2 = 9$, $3^3 = 27$, $3^4 = 81$, $3^5 = 243, \ldots$ we see that the final digits cycle through the four numbers 3, 9, 7, 1. As $2011 = 502 \times 4 + 3$, the last digit of 3^{2011} is 7.

5. **B** As the sum of the angles in a triangle is 180° and all four angles in a rectangle are 90°, the sum of the two marked angles in the triangle is $180° - 90° = 90°$.

Each interior angle of a regular hexagon is 120° and the sum of the angles in a quadrilateral is 360°; hence the sum of the two marked angles in the quadrilateral is $360° - 90° - (360° - 120°) = 30°$.

Hence the sum of the four marked angles is $90° + 30° = 120°$.

6. **C** Let Granny's age today be G and Gill's age today be g.

Therefore $G = 15g \ldots(1)$ and $G + 4 = (g + 4)^2 \ldots(2)$.

Substituting (1) into (2) gives $15g + 4 = g^2 + 8g + 16$, hence $g^2 - 7g + 12 = 0$.

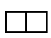
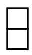
Thus $(g - 3)(g - 4) = 0$, hence $g = 3$ or 4.

As G is even and $G = 15g$, g is also even. Thus $g = 4$ and $G = 15 \times 4 = 60$.

Hence today, Granny is 56 years older than Gill.

7. **D** In order to form a triangle, x must exceed the difference between 4 and 5 and x must be less than the sum of 4 and 5, i.e. $1 < x < 9$.

Hence $x = 2, 3, 4, 5, 6, 7$ or 8. So x can have 7 different values.

8. **C** The 1×2 rectangles can appear in two different ways: A  or B .

If, in the given shape, A forms the top 1×2 rectangle then the possible different ways to fill the remaining 2×4 rectangle are, from left to right:

A, A, A, A; A, A, B, B; B, A, A, B; B, B, A, A; B, B, B, B.

If A does not form the top 1×2 rectangle, then the only possible way is to use 4B and an A. Hence there are 6 ways of dividing the given shape into 1×2 rectangles.

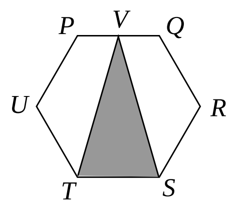
9. **B** Let the centre cube in the $3 \times 3 \times 3$ block be red. As no cubes of the same colour meet face-to-face then the 6 centre cubes on the outer faces must be yellow. All six outer faces are as shown.

Y	R	Y
R	Y	R
Y	R	Y

Thus 14 faces are yellow and 13 faces are red. If the centre cube is yellow then the situation is reversed. Hence the difference between the largest number of red cubes that Sam can use and the smallest number is 1.

10. **C** The area of a triangle is $\frac{1}{2}ab \sin C$. The maximum area is attained when $\angle C = 90°$. Hence, in order to maximise the area, the triangle must be right-angled with common side lengths equal to 5. Let x be the side length of the hypotenuse, so, by Pythagoras' Theorem, $x^2 = 5^2 + 5^2 = 50$. Thus $x = 5\sqrt{2}$ is the length that should be chosen.

11. **C** Let x be the side length of the regular hexagon $PQRSTU$ and let $h = PT = QS$, the perpendicular height of triangle STV.



Thus the area of triangle STV is $\frac{1}{2}xh$ and the areas of triangles PTV and QSV are both $\frac{1}{2}\left(\frac{1}{2}xh\right) = \frac{1}{4}xh$. The perpendicular heights of triangles PTU and QRS are

$$\frac{UR - PQ}{2} = \frac{2x - x}{2} = \frac{x}{2}.$$

Hence the area of each of triangles PTU and QRS is $\frac{1}{2}h \times \frac{1}{2}x = \frac{1}{4}hx$.

Therefore the area of triangle STV is one third of the area of $PQRSTU$.

12. **D** The primorial of 7 is $2 \times 3 \times 5 \times 7 = 210$. As 8, 9 and 10 are not prime numbers, they also have a primorial of 210. The primorial of 11 is $2 \times 3 \times 5 \times 7 \times 11 = 2310$.

Hence there are exactly four different whole numbers which have a primorial of 210.

13. **D** Let the centres of the starting and finishing squares in the maze have coordinates (1,4) and (4,1) respectively. Each path must pass through (2,3) and (3,2). There are two different routes from (1,4) to (2,3). The next visit is to (3,3) or (2,2).

When visiting (3,3) the next visit has to be (3,2) as (3,4), (4,3) and (4,4) cannot be visited without subsequently revisiting a square. From (2,2) the next valid visit is to (1,2), (2,1) or (3,2). From each of these points there is only one route to (3,2). Thus there are four ways of visiting (3,2). Upon visiting (3,2), the only valid route through the maze is (4,2) then (4,1).

Hence the number of different routes through the maze is $2 \times 4 = 8$.

14. **C** Let us define T_n to represent an equilateral triangle with side length n cm. Then an equilateral triangle of side length 4 cm can be divided into smaller equilateral triangles as follows:

$1 \times T_3$ and $7 \times T_1$	$4 \times T_2$	$3 \times T_2$ and $4 \times T_1$
$2 \times T_2$ and $8 \times T_1$	$1 \times T_2$ and $12 \times T_1$	$16 \times T_1$.

The number of triangles used are: 8, 4, 7, 10, 13 and 16. So it is not possible to dissect the original triangle into 12 triangles.

15. B If a, b are roots of $x^2 + ax + b = 0$ then $x^2 + ax + b = 0$ must be $(x - a)(x - b) = 0$. As $(x - a)(x - b) = x^2 + (-a - b)x + ab$ then $a = -a - b$ and $b = ab$. If $b = 0$ we see immediately that $a = 0$. But this is not possible as a and b are different. If $b \neq 0$ then $a = 1$ and $b = -2$. So there is just one solution pair.

16. E Let $QR = x$ and $RS = y$ in the rectangle $PQRS$. Hence the area of $PQRS$ is xy.

The area of triangle QRT is $\frac{1}{2}RT \times x = \frac{1}{5}xy$, hence $RT = \frac{2}{5}y$. Thus $TS = RS - RT = \frac{3}{5}y$.

The area of triangle TSU is $\frac{1}{2}SU \times \frac{3}{5}y = \frac{1}{8}xy$, hence $SU = \frac{5}{12}x$.

Therefore the area of triangle PUQ is $\frac{1}{2} \times \frac{7}{12}xy = \frac{7}{24}xy$.

Hence, as a fraction of the area of rectangle $PQRS$, the area of triangle QTU is

$$\frac{xy\left(1 - \frac{1}{5} - \frac{1}{8} - \frac{7}{24}\right)}{xy} = \frac{23}{60}.$$

17. A Let a, b, o and p represent the percentage of pupils liking apples, bananas, oranges and pears respectively.

As $a = 85$, there are 15% of pupils who do not like apples. As $b = 80$, $a \cap b$ is greater than or equal to $80 - (100 - 85) = 65$. As $o = 75$, $a \cap b \cap o$ is greater than or equal to $75 - (100 - 65) = 40$. Finally, as $p = 70$, $a \cap b \cap o \cap p$ is greater than or equal to $70 - (100 - 40) = 10$.

Hence the percentage of pupils who like all four fruits is at least 10%.

18. E Multiplying $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$ throughout by $2xy$ gives $2y + 2x = xy$. Hence

$$xy = 2(x + y). \qquad (1)$$

But since $x^2y + xy^2 = xy(x + y)$, we can use (1) to give $xy(x + y) = 2(x + y)(x + y)$.

But $x + y = 20$, hence $x^2y + xy^2 = 2 \times 20^2 = 800$.

19. B As each square has area 1 its side length must be 1.

The external angle of the small regular octagon is $\frac{1}{8} \times 360° = 45°$.

Hence, as the sum of the angles on a straight line is 180° and the sum of the angles in a kite is 360°, the four angles in each of the eight kites (white) are: 90°, 90°, 135° and 45°. As the light grey kites and the white kites are similar, the interior angles are the same. Two of the sides of the grey kite have length 1. Let the other sides have length a. Using the Cosine Rule twice within a light grey kite, the square of the short diagonal is $1^2 + 1^2 - 2 \times 1 \times 1\cos 135° = a^2 + a^2 - 2a \times a\cos 45°$. Hence

$$2 + 2 \times \frac{1}{\sqrt{2}} = 2a^2 - 2a^2 \times \frac{1}{\sqrt{2}}.$$

Thus $a^2 = \dfrac{1 + \frac{1}{\sqrt{2}}}{1 - \frac{1}{\sqrt{2}}} = \dfrac{\sqrt{2} + 1}{\sqrt{2} - 1}$ and so $a = \sqrt{2} + 1$.

But the area of one of the light grey kites is $2 \times \frac{1}{2}a \times 1 = a$.

Hence the area of one of the light grey kites is $\sqrt{2} + 1$.

20. **B** Squaring the equation $\sqrt{x} - \sqrt{11} = \sqrt{y}$ gives $x - 2\sqrt{11x} + 11 = y \ldots (1)$. You see here that $2\sqrt{11x}$ is an integer. Thus $x = 11a^2$ for some integer a. Hence in (1), $y = 11a^2 - 22a + 11 = 11(a^2 - 2a + 1)$. Thus $\frac{x}{y} = \left(\frac{a}{a-1}\right)^2$ whose maximum value, for integer a, is easily seen to be $\left(\frac{2}{1}\right)^2 = 4$.

21. **C** At least one of d'Artagnan and Athos is lying. One of Porthos or Aramis is telling the truth and the other is lying. So the number of liars is either two (d'Artagnan and Porthos) or three (all except Porthos).

22. **A** As the sum of the angles in a triangle is 180°, in triangle CBF, $\angle BFC = 90°$. As vertically opposite angles are equal $\angle DFE = \angle BFC = 90°$. As the sum of the angles on a straight line is 180°, $\angle DFB = \angle EFC = 90°$. Hence in triangle EFD, $\tan\alpha = \frac{DF}{EF}$: in triangle DFB, $\tan 10° = \frac{DF}{FB}$: in triangle BFC, $\tan 20° = \frac{FB}{FC}$ and in triangle CEF, $\tan 50° = \frac{EF}{FC}$. Thus

$$\tan\alpha = \frac{DF}{EF} = \frac{\tan 10° FB}{EF} = \frac{\tan 10° \tan 20° FC}{EF} = \frac{\tan 10° \tan 20°}{\tan 50°}.$$

23. **B** $x^2 + y^2 + 2xy + 6x + 6y + 4 = (x+y)^2 + 6(x+y) + 4 = [(x+y)+3][(x+y)+3] - 5 = (x + y + 3)^2 - 5$. But $(x + y + 3)^2 \geqslant 0$ for all values of x and y. As $x + y + 3$ can be 0 for appropriate values of x, y the minimum value of $x^2 + y^2 + 2xy + 6x + 6y + 4$ is -5.

24. **B** Let the radii of the circles from smallest to largest be r_1, r_2 and r_3 respectively. Hence $16r_1 = r_3 + 2r_2 + r_1$, thus $r_3 = 15r_1 - 2r_2 \ldots (1)$. Let $r_1 + x$ be the distance from Q to the centre of the smallest circle. By similar triangles,

$$\frac{r_1}{r_1 + x} = \frac{r_2}{x + 2r_1 + r_2} = \frac{r_3}{16r_1 + r_1 + x} \ldots (2).$$

Thus $r_1(x + 2r_1 + r_2) = r_2(r_1 + x)$. Hence $r_2 = \frac{r_1 x + 2r_1^2}{x} \ldots (3)$. From (1) and (2) $\frac{r_1 x}{r_1 + x} = \frac{(15r_1 - 2r_2)x}{17r_1 + x}$ hence $\frac{r_1 x}{r_1 + x} = \frac{15r_1 x - 2(r_1 x + 2r_1^2)}{17r_1 + x}$.
Dividing throughout by r_1 and simplifying gives $12x^2 - 8r_1 x - 4r_1^2 = 0$.
Hence $(3x + r_1)(x - r_1) = 0$ so, as $r_1 > 0$, $x = r_1$. Thus
$\sin\frac{\angle PQR}{2} = \frac{r_1}{r_1 + x} = \frac{r_1}{2r_1} = \frac{1}{2}$. Hence $\frac{1}{2}\angle PQR = 30°$ so $\angle PQR = 60°$.

25. **C** Three vertices of the smaller cube lie on edges of the larger cube, the same distance along each. Let this distance be x and let the distance between any two of these vertices be y. Hence, by Pythagoras' Theorem, $y^2 = x^2 + x^2$ and, as the side length of the smaller cube is 3, $y^2 = 3^2 + 3^2$. Thus $x = 3$ and $y = 3\sqrt{2}$.

The intersection of the cubes forms two congruent tetrahedra of base area equal to $\frac{1}{2}y^2 \sin 60° = \frac{1}{2}(3\sqrt{2})^2 \times \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{2}$. Let h be the perpendicular height of the tetrahedra. Hence, using Pythagoras' Theorem twice gives $9 = h^2 + 6$, thus $h = \sqrt{3}$.

Thus the total volume of the sculpture is $4^3 + 3^3 - 2 \times \frac{1}{3} \times \frac{9\sqrt{3}}{2} \times \sqrt{3} = 91 - 9 = 82$.