Question 1

Carry out the following integrations:

1. $\displaystyle\int x e^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$

2. $\displaystyle\int 3x\cos 2x\,dx = \frac{3}{2}x\sin 2x + \frac{3}{4}\cos 2x + C$

3. $\displaystyle\int x\sin 4x\,dx = -\frac{1}{4}x\cos 4x + \frac{1}{16}\sin 4x + C$

4. $\displaystyle\int -2x\sin 5x\,dx = \frac{2}{5}x\cos 5x - \frac{2}{25}\sin 5x + C$

5. $\displaystyle\int (1-2x)e^{-x}\,dx = (2x-1)e^{-x} + 2e^{-x} + C$

1.
$$\begin{aligned}\int x e^{2x}\,dx &= \tfrac{1}{2}xe^{2x} - \int \tfrac{1}{2}e^{2x}\,dx \\ &= \tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x} + C\end{aligned}$$
$$\begin{cases} x \to 1 \\ \tfrac{1}{2}e^{2x} \to e^{2x}\end{cases}$$

2.
$$\begin{aligned}\int 3x\cos 2x\,dx &= \tfrac{3}{2}x\sin 2x - \int \tfrac{3}{2}\sin 2x\,dx \\ &= \tfrac{3}{2}x\sin 2x - \left(-\tfrac{3}{4}\cos 2x\right) + C \\ &= \tfrac{3}{2}x\sin 2x + \tfrac{3}{4}\cos 2x + C\end{aligned}$$
$$\begin{cases} 3x \to 3 \\ \tfrac{1}{2}\sin 2x \to \cos 2x\end{cases}$$

3.
$$\begin{aligned}\int x\sin 4x\,dx &= -\tfrac{1}{4}x\cos 4x - \int -\tfrac{1}{4}\cos 4x\,dx \\ &= -\tfrac{1}{4}x\cos 4x + \int \tfrac{1}{4}\cos 4x\,dx \\ &= -\tfrac{1}{4}x\cos 4x + \tfrac{1}{16}\sin 4x + C\end{aligned}$$
$$\begin{cases} x \to 1 \\ -\tfrac{1}{4}\cos 4x \to \sin 4x\end{cases}$$

4.
$$\begin{aligned}\int -2x\sin 5x\,dx &= \tfrac{2}{5}x\cos 5x - \int \tfrac{2}{5}\cos 5x\,dx \\ &= \tfrac{2}{5}x\cos 5x - \left(\tfrac{2}{25}\sin 5x\right) + C \\ &= \tfrac{2}{5}x\cos 5x - \tfrac{2}{25}\sin 5x + C\end{aligned}$$
$$\begin{cases} -2x \to -2 \\ -\tfrac{1}{5}\cos 5x \to \sin 5x\end{cases}$$

5.
$$\begin{aligned}\int (1-2x)e^{-x}\,dx &= -(1-2x)e^{-x} - \int 2e^{-x}\,dx \\ &= (2x-1)e^{-x} - \left(-2e^{-x}\right) + C \\ &= (2x-1)e^{-x} + 2e^{-x} + C\end{aligned}$$
$$\begin{cases} 1-2x \to -2 \\ -e^{-x} \to e^{-x}\end{cases}$$

Question 4

Carry out each of the following integrations.

1. $\displaystyle\int \frac{x}{\left(x^2-1\right)^3}\,dx = -\frac{1}{4}\left(x^2-1\right)^{-2} + C$

2. $\displaystyle\int \cos x \sin x\,dx = \frac{1}{2}\sin^2 x + C = -\frac{1}{2}\cos^2 x + C = -\frac{1}{4}\cos 2x + C$

3. $\displaystyle\int \frac{4x}{\sqrt{1-2x^2}}\,dx = -2\sqrt{1-2x^2} + C$

4. $\displaystyle\int \sec^2 x\left(1+\tan^2 x\right)dx = \tan x + \frac{1}{3}\tan^3 x + C$

5. $\displaystyle\int \sec^2 x\left(1+\tan x\right)dx = \frac{1}{2}\left(1+\tan x\right)^2 + C$

1. $\displaystyle\int \frac{x}{(x^2-1)^3}\,dx = \int x(\underbrace{x^2-1}_{2x})^{-3}\,dx = -\frac{1}{4}(x^2-1)^{-2} + C$

2. $\displaystyle\int \cos x \sin x\,dx = \frac{1}{2}\sin^2 x + C$ since $\frac{d}{dx}(\sin x) = \cos x$

   or $-\frac{1}{2}\cos^2 x + C$ since $\frac{d}{dx}(\cos x) = -\sin x$

3. $\displaystyle\int \frac{4x}{\sqrt{1-2x^2}}\,dx = \int 4x(\underbrace{1-2x^2}_{-4x})^{-\frac{1}{2}}\,dx = -2(1-2x^2)^{\frac{1}{2}} + C$

4. $\displaystyle\int \sec^2 x(1+\tan^2 x)\,dx = \int \sec^2 x + \sec^2 x \tan^2 x\,dx = \tan x + \frac{1}{3}\tan^3 x + C$

   $\frac{d}{dx}(\tan x) = \sec^2 x$

5. $\displaystyle\int \sec^2 x(1+\tan x)\,dx = \int \sec^2 x + \sec^2 x \tan x\,dx = \tan x + \frac{1}{2}\tan^2 x + C$

   $\frac{d}{dx}(\tan x) = \sec^2 x$

   or $\displaystyle\int \sec^2 x(1+\tan x)\,dx = \frac{1}{2}(1+\tan x)^2 + C$

Question 2

Carry out the following integrations **by substitution** only.

1. $$\int 6x(3x-1)^3\,dx = \frac{2}{15}(3x-1)^5 + \frac{1}{6}(3x-1)^4 + C$$

2. $$\int \frac{5x}{5x-1}\,dx = \frac{1}{5}(5x-1) + \frac{1}{5}\ln|5x-1| + C$$

3. $$\int 3x\left(x^2+1\right)^{\frac{1}{2}}\,dx = \left(x^2+1\right)^{\frac{3}{2}} + C$$

4. $$\int \frac{3x^2}{2x^3+1}\,dx = \frac{1}{2}\ln\left|2x^3+1\right| + C$$

1. $$\int 6x(3x-1)^3\,dx = \int 6x\,u^3\,\frac{du}{3} = \frac{1}{3}\int (2u+2)u^3\,du =$$
$$\frac{2}{3}\int u^4 + u^3\,du = \frac{2}{3}\left[\frac{1}{5}u^5 + \frac{1}{4}u^4\right] = \frac{2}{15}u^5 + \frac{1}{6}u^4 + C$$
$$= \frac{2}{15}(3x-1)^5 + \frac{1}{6}(3x-1)^4 + C$$

$u = 3x-1$, $\frac{du}{dx} = 3$, $dx = \frac{du}{3}$; $3x = u+1$, $6x = 2u+2$

2. $$\int \frac{5x}{5x-1}\,dx = \int \frac{5x}{u} \times \frac{du}{5} = \int \frac{u+1}{u} \times \frac{du}{5}$$
$$= \frac{1}{5}\int 1 + \frac{1}{u}\,du = \frac{1}{5}\left[u + \ln|u|\right] = \frac{1}{5}(5x-1) + \frac{1}{5}\ln|5x-1| + C$$

$u = 5x-1$, $\frac{du}{dx} = 5$, $dx = \frac{du}{5}$; $5x = u+1$

3. $$\int 3x(x^2+1)^{\frac{1}{2}}\,dx = \int 3\cancel{x}\,u^{\frac{1}{2}}\,\frac{du}{2\cancel{x}} = \int \frac{3}{2}u^{1/2}\,du$$
$$\frac{3/2}{3/2}u^{3/2} + C = (x^2+1)^{3/2} + C$$

$u = x^2+1$, $\frac{du}{dx} = 2x$, $dx = \frac{du}{2x}$

4. $$\int \frac{3x^2}{2x^3+1}\,dx = \int \frac{3\cancel{x^2}}{u} \times \frac{du}{6\cancel{x^2}} = \frac{1}{2}\int \frac{1}{u}\,du$$
$$= \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|2x^3+1\right| + C$$

$u = 2x^3+1$, $\frac{du}{dx} = 6x^2$, $dx = \frac{du}{6x^2}$

5. $$\int x(2x-1)^5\,dx = \int x\,u^5\,\frac{du}{\ldots} = \int \frac{u+1}{2} \times u^5 \times \frac{du}{\ldots}$$

$u = 2x-1$

2. $$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx = \frac{1}{4}\left(\sqrt{3}-1\right), \quad \text{use } x = 2\cos\theta$$

$$\int_1^{\sqrt{2}} \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{1}{(2\cos\theta)^2\sqrt{4-(2\cos\theta)^2}}(-2\sin\theta\,d\theta)$$

$$= \int_{\frac{\pi}{3}}^{\frac{\pi}{4}} \frac{-2\sin\theta}{4\cos^2\theta\sqrt{4-4\cos^2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{4\cos^2\theta\sqrt{4(1-\cos^2\theta)}}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{4\cos^2\theta\sqrt{4\sin^2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{2\sin\theta}{8\cos^2\theta\sin\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{4}\sec^2\theta\,d\theta$$

$$= \left[\frac{1}{4}\tan\theta\right]_{\pi/4}^{\pi/3} = \frac{1}{4}\left(\tan\frac{\pi}{3}-\tan\frac{\pi}{4}\right) = \frac{1}{4}\left(\sqrt{3}-1\right)$$

$x = 2\cos\theta$

$\frac{dx}{d\theta} = -2\sin\theta$

$dx = -2\sin\theta\,d\theta$

- $x=1$ $\quad 1 = 2\cos\theta$, $\frac{1}{2}=\cos\theta$, $\theta=\frac{\pi}{3}$
- $x=\sqrt{2}$ $\quad \sqrt{2}=2\cos\theta$, $\frac{\sqrt{2}}{2}=\cos\theta$, $\theta=\frac{\pi}{4}$

3. $\displaystyle\int_0^1 \frac{1}{\left(1+x^2\right)^2}\,dx = \frac{1}{8}(\pi+2)$, use $x=\tan\theta$

$$\int_0^1 \frac{1}{(1+x^2)^2}\,dx = \int_0^{\frac{\pi}{4}} \frac{1}{(1+\tan^2\theta)^2}\sec^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2\theta}\,d\theta = \int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}+\frac{1}{2}\cos 2\theta\,d\theta = \left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$

$$= \left(\frac{\pi}{8}+\frac{1}{4}\right)-\left(0+0\right) = \frac{\pi}{8}+\frac{1}{4} = \frac{1}{8}(\pi+2)$$

$x=\tan\theta$

$\frac{dx}{d\theta}=\sec^2\theta$

$dx = \sec^2\theta\,d\theta$

$x=0$, $\tan\theta=0$, $\theta=0$

$x=1$, $\tan\theta=1$, $\theta=\frac{\pi}{4}$

4. $\displaystyle\int_{\sqrt{2}}^2 \frac{1}{x^2\sqrt{x^2-1}}\,dx = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$, use $x=\sec\theta$

$$\int_{\sqrt{2}}^{2} \frac{1}{x^2\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta\sqrt{\sec^2\theta-1}}(\sec\theta\tan\theta)\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta\tan\theta}{\sec^2\theta\sqrt{\tan^2\theta}}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cancel{\sec\theta}\,\cancel{\tan\theta}}{\sec^{\cancel{2}}\theta\,\cancel{\tan\theta}}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec\theta}\,d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos\theta\,d\theta = \Big[\sin\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \sin\frac{\pi}{3} - \sin\frac{\pi}{4} = \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} = \frac{1}{2}\left(\sqrt{3}-\sqrt{2}\right)$$

$x = \sec\theta$

$\frac{dx}{d\theta} = \sec\theta\tan\theta$

$dx = \sec\theta\tan\theta\,d\theta$

$x=2 \quad 2 = \sec\theta$

$\frac{1}{2} = \cos\theta$

$\theta = \pi/3$

$x=\sqrt{2} \quad \sqrt{2} = \sec\theta$

$\frac{1}{\sqrt{2}} = \cos\theta$

$\theta = \pi/4$

Question 2

Integrate:

1. $\int (2+\sin x)^2\, dx$

2. $\int \sin x\left(1+\sec^2 x\right) dx$

3. $\int (1-2\cos x)^2\, dx$

4. $\int \frac{1}{\cos^2 x\tan^2 x}\, dx$

5. $\int 2+2\tan^2 x\, dx$

1. $\int (2+\sin x)^2\,dx = \int 4 + 4\sin x + \sin^2 x\,dx =$

$= \int 4 + 4\sin x + \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx = \int \frac{9}{2} + 4\sin x - \frac{1}{2}\cos 2x\,dx$

$= \frac{9}{2}x - 4\cos x - \frac{1}{4}\sin 2x + C$

2. $\int \sin x\,(1+\sec^2 x)\,dx = \int \sin x + \sin x\sec^2 x\,dx = \int \sin x + \frac{\sin x}{\cos^2 x}\,dx$

$\int \sin x + \frac{\sin x}{\cos x}\times\frac{1}{\cos x}\,dx = \int \sin x + \tan x\sec x\,dx = -\cos x + \sec x + C$

3. $\int (1-2\cos x)^2\,dx = \int 1 - 4\cos x + 4\cos^2 x\,dx = \int 1 - 4\cos x + 4\left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)dx$

$= \int 1 - 4\cos x + 2 + 2\cos 2x\,dx = \int 3 - 4\cos x + 2\cos 2x\,dx$

$= 3x - 4\sin x + \sin 2x + C$

4. $\int \frac{1}{\cos^2 x\tan^2 x}\,dx = \int \frac{1}{\cos^2 x \times \frac{\sin^2 x}{\cos^2 x}}\,dx = \int \operatorname{cosec}^2 x\,dx = -\cot x + C$

5. $\int 2 + 2\tan^2 x\,dx = \int 2 + 2(\sec^2 x - 1)\,dx = \int 2\sec^2 x\,dx = 2\tan x + C$

Question 5

Carry out each of the following integrations.

1. $$\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,dx = -\frac{2}{2x-1} - \frac{1}{3}\ln|2-3x| - \frac{1}{2}\ln|2x-1| + C$$

2. $$\int \frac{1}{x^2(x-1)}\,dx = \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$$

3. $$\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\,dx = 5\ln|x-3| + 3\ln|x+1| + \frac{4}{x+1} + C$$

4. $$\int \frac{1}{x(x-2)}\,dx = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$$

5. $$\int \frac{1}{x^2-4}\,dx = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$$

1. $\displaystyle\int \frac{10x^2-23x+11}{(2-3x)(2x-1)^2}\,dx$ = BY PARTIAL FRACTIONS

$\displaystyle = \int \frac{1}{2-3x} + 4(2x-1)^{-2} - \frac{1}{2x-1}\,dx$

$= -\frac{1}{3}\ln|2-3x| - 2(2x-1)^{-1} - \frac{1}{2}\ln|2x-1| + C$

$= -\frac{1}{3}\ln|2-3x| - \frac{2}{2x-1} - \frac{1}{2}\ln|2x-1| + C$

$\dfrac{10x^2-23x+11}{(2-3x)(2x-1)^2} \equiv \dfrac{A}{2-3x} + \dfrac{B}{(2x-1)^2} + \dfrac{C}{2x-1}$

$10x^2-23x+11 \equiv A(2x-1)^2 + B(2-3x) + C(2x-1)(2-3x)$

- If $x=\frac{1}{2} \Rightarrow \frac{5}{2} - \frac{23}{2} + 11 = \frac{1}{2}B \Rightarrow \boxed{B=4}$
- If $x=\frac{2}{3} \Rightarrow \frac{40}{9} - \frac{46}{3} + 11 = \frac{1}{9}A \Rightarrow \boxed{A=1}$
- If $x=0 \Rightarrow \left.\begin{array}{l} 11 = A+2B-2C \\ 11 = 1+8-2C \end{array}\right\} \Rightarrow \boxed{C=-1}$

2. $\displaystyle\int \frac{1}{x^2(x-1)}\,dx$ = BY PARTIAL FRACTIONS

$\displaystyle = \int -x^{-2} - \frac{1}{x} + \frac{1}{x-1}\,dx$

$= x^{-1} - \ln|x| + \ln|x-1| + C$

$= \frac{1}{x} + \ln\left|\frac{x-1}{x}\right| + C$

$\dfrac{1}{x^2(x-1)} \equiv \dfrac{A}{x^2} + \dfrac{B}{x} + \dfrac{C}{x-1}$

$1 \equiv A(x-1) + Bx(x-1) + Cx^2$

- If $x=1 \Rightarrow \boxed{1=C}$
- If $x=0 \Rightarrow 1=-A \Rightarrow \boxed{A=-1}$
- If $x=2 \Rightarrow \left.\begin{array}{l} 1 = A+2B+4C \\ 1 = -1+2B+4 \end{array}\right\} \Rightarrow \boxed{B=-1}$

3. $\displaystyle\int \frac{8(x^2+1)}{(x-3)(x+1)^2}\,dx$ = BY PARTIAL FRACTIONS

$\displaystyle = \int \frac{5}{x-3} - 4(x+1)^{-2} + \frac{3}{x+1}\,dx$

$= 5\ln|x-3| + 4(x+1)^{-1} + 3\ln|x+1| + C$

$= 5\ln|x-3| + 3\ln|x+1| + \frac{4}{x+1} + C$

$\dfrac{8(x^2+1)}{(x-3)(x+1)^2} \equiv \dfrac{A}{x-3} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+1}$

$8(x^2+1) \equiv A(x+1)^2 + B(x-3) + C(x+1)(x-3)$

- If $x=-1 \Rightarrow 16 = -4B \Rightarrow \boxed{B=-4}$
- If $x=3 \Rightarrow 80 = 16A \Rightarrow \boxed{A=5}$
- If $x=0 \Rightarrow \left.\begin{array}{l} 8 = A-3B-3C \\ 8 = 5+12-3C \end{array}\right\} \Rightarrow \boxed{C=3}$

4. $\displaystyle\int \frac{1}{x(x-2)}\,dx$ = BY PARTIAL FRACTIONS

$\displaystyle \int \frac{-1/2}{x} + \frac{1/2}{x-2}\,dx = -\frac{1}{2}\ln|x| + \frac{1}{2}\ln|x-2| + C$

$= \frac{1}{2}\left[\ln|x-2| - \ln|x|\right] + C = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$

$\dfrac{1}{x(x-2)} \equiv \dfrac{A}{x} + \dfrac{B}{x-2}$

$1 \equiv A(x-2) + Bx$

If $x=0 \Rightarrow 1=-2A \Rightarrow A=-\frac{1}{2}$

If $x=2 \Rightarrow 1=2B \Rightarrow B=\frac{1}{2}$

5. $\displaystyle\int \frac{1}{x^2-4}\,dx = \int \frac{1}{(x+2)(x-2)}\,dx$ = BY PARTIAL FRACTIONS

$\displaystyle = \int \frac{1/4}{x-2} - \frac{1/4}{x+2}\,dx = \frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C$

$= \frac{1}{4}\left[\ln|x-2| - \ln|x+2|\right] + C = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right| + C$

$\dfrac{1}{(x+2)(x-2)} \equiv \dfrac{A}{x+2} + \dfrac{B}{x-2}$

$1 \equiv A(x-2) + B(x+2)$

- If $x=2 \quad 4B=1 \Rightarrow B=\frac{1}{4}$
- If $x=-2 \quad -4A=1 \Rightarrow A=-\frac{1}{4}$

Differential equations

Question 2 (**+)

Water is draining out of a tank so that the height of the water, h m, in time t minutes, satisfies the differential equation

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant.

The initial height of the water is 2.25 m and 20 minutes later it drops to 1 m.

a) Show that the solution of the differential equation can be written as

$$h = \frac{(60-t)^2}{1600}.$$

b) Find after how long the height of the water drops to 0.25 m.

(a)

$$\frac{dh}{dt} = -kh^{\frac{1}{2}}$$

$$\Rightarrow dh = -kh^{\frac{1}{2}}\,dt$$

$$\Rightarrow \frac{1}{h^{\frac{1}{2}}}\,dh = -k\,dt$$

$$\Rightarrow \int h^{-\frac{1}{2}}\,dh = \int -k\,dt$$

$$\Rightarrow 2h^{\frac{1}{2}} = -kt + C$$

$$\Rightarrow \boxed{h^{\frac{1}{2}} = A - Bt}$$

when $t=0$ $h=2.25$

$$\sqrt{2.25} = A$$

$$A = \frac{3}{2}$$

$$\Rightarrow \boxed{h^{\frac{1}{2}} = \frac{3}{2} - Bt}$$

when $t=20$ $h=1$

$$1 = \frac{3}{2} - 20B$$

$$2 = 3 - 40B$$

$$B = \frac{1}{40}$$

$$\therefore h^{\frac{1}{2}} = \frac{3}{2} - \frac{1}{40}t$$

$$h^{\frac{1}{2}} = \frac{60-t}{40}$$

$$h = \frac{(60-t)^2}{1600}$$

// AS REQUIRED

(b) when $h=0.25$

$$0.25 = \frac{(60-t)^2}{1600}$$

$$400 = (60-t)^2$$

$$20 = 60 - t$$

$$t = 40$$

Question 4 (**+)

An entomologist believes that the population P insects in a colony, t weeks after it was first observed, obeys the differential equation

$$\frac{dP}{dt} = kP^2,$$

where k is a positive constant.

Initially 1000 insects were observed, and this population doubled after 4 weeks.

a) Find a solution of the differential equation, in the form $P = f(t)$.

b) Give <u>two different</u> reasons why the model can only work for small values of t.

$$\boxed{P = \frac{8000}{8-t}}$$

(a) $\frac{dP}{dt} = kP^2$

$\Rightarrow \frac{1}{P^2}\,dP = k\,dt$

$\Rightarrow \int P^{-2}\,dP = \int k\,dt$

$\Rightarrow -P^{-1} = kt + C$

$\Rightarrow -\frac{1}{P} = kt + C$

$\Rightarrow \boxed{\frac{1}{P} = At + B}$

- when $t=0$ $P=1000$

$\boxed{\frac{1}{1000} = B}$

$\therefore \boxed{\frac{1}{P} = At + \frac{1}{1000}}$

- $t=4$ $P=2000$

$\frac{1}{2000} = 4A + \frac{1}{1000}$

$4A = -\frac{1}{2000}$

$\boxed{A = -\frac{1}{8000}}$

$\therefore \frac{1}{P} = \frac{1}{1000} - \frac{t}{8000}$

$\frac{1}{P} = \frac{8-t}{8000}$

$P = \frac{8000}{8-t}$ //

(b) If $t=8$ P becomes infinite

if $t>8$ P becomes negative //

Question 13 (***+)

A population P, in millions, at a given time t years, satisfies the differential equation

$$\frac{dP}{dt} = P(1-P).$$

Initially the population is one quarter of a million.

a) Solve the differential equation to show that

$$\frac{3P}{1-P} = e^{t}.$$

b) Show further that

$$P = \frac{1}{1+3e^{-t}}.$$

c) Show mathematically that the limiting value for this population is one million.

d) Find, to two decimal places, the time it takes for the population to reach three quarters of its limiting value.

$\boxed{}$, $\boxed{t \to \infty,\ P \to 1}$, $\boxed{t = \ln 9 \simeq 2.20}$

(a)

$$\frac{dP}{dt} = P(1-P)$$

$$\Rightarrow \frac{1}{P(1-P)}\,dP = 1\,dt$$

$$\Rightarrow \int \frac{1}{P(1-P)}\,dP = \int 1\,dt$$

$$\Rightarrow \int \frac{1}{P} + \frac{1}{1-P}\,dP = \int 1\,dt$$

$$\Rightarrow \ln P - \ln|1-P| = t + C$$

$$\Rightarrow \ln\left|\frac{P}{1-P}\right| = t + C$$

$$\Rightarrow \frac{P}{1-P} = e^{t+C}$$

$$\Rightarrow \boxed{\frac{P}{1-P} = Ae^{t}} \quad (A = e^{C})$$

BY PARTIAL FRACTIONS

$$\frac{1}{P(1-P)} \equiv \frac{A}{P} + \frac{B}{1-P}$$

$$1 \equiv A(1-P) + BP$$

If $P=0$ $A=1$

If $P=1$ $B=1$

when $t=0$ $P=\frac{1}{4}$

$$\frac{\frac{1}{4}}{1-\frac{1}{4}} = Ae^{0}$$

$$\frac{1}{3} = A$$

$$\therefore \frac{P}{1-P} = \frac{1}{3}e^{t}$$

$$\frac{3P}{1-P} = e^{t}$$

// AS REQUIRED

(b)

$$3P = (1-P)e^{t}$$

$$3P = e^{t} - Pe^{t}$$

$$3P + Pe^{t} = e^{t}$$

$$P(3+e^{t}) = e^{t}$$

$$P = \frac{e^{t}}{3+e^{t}}$$

MULTIPLY TOP/BOTTOM BY e^{-t}

$$P = \frac{e^{-t}e^{t}}{3e^{-t}+e^{-t}e^{t}}$$

$$P = \frac{1}{3e^{-t}+1}$$

// AS REQUIRED

(c) AS $t \to \infty$ $e^{-t} \to 0$

$\therefore P \to 1$

IE POPULATION TENDS TO ONE MILLION //

(d) LIMITING VALUE IS 1

$\therefore$ WE NEED t WHEN $P = \frac{3}{4}$

USING $\frac{3P}{1-P} = e^{t}$

$$\frac{3\times\frac{3}{4}}{1-\frac{3}{4}} = e^{t}$$

$$e^{t} = 9$$

$$t = \ln 9$$

$$t \approx 2.20$$

//

Question 11 (***+)

Find a general solution of the differential equation

$$\sec 3x \frac{dy}{dx} = \cot^2 2y$$

giving the answer in the form $f(x, y) = c$.

$\boxed{3\tan 2y - 6y - 2\sin 3x = C}$

$$\sec 3x \frac{dy}{dx} = \cot^2 2y$$

$$\Rightarrow \frac{1}{\cot^2 2y}\,dy = \frac{1}{\sec 3x}\,dx$$

$$\Rightarrow \int \tan^2 2y\,dy = \int \cos 3x\,dx$$

$$\Rightarrow \int \sec^2 2y - 1\,dy = \int \cos 3x\,dx$$

$$\Rightarrow \frac{1}{2}\tan 2y - y = \frac{1}{3}\sin 3x + C$$

$$\Rightarrow 3\tan 2y - 6y = 2\sin 3x + C$$

$$\Rightarrow 3\tan 2y - 6y - 2\sin 3x = C$$

Question 13 (****)

Show that a general solution of the differential equation

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

is given by

$$y = \frac{Ae^x - 3}{2Ae^x - 1},$$

where A is an arbitrary constant.

$\boxed{\text{proof}}$

$$5\frac{dy}{dx} = 2y^2 - 7y + 3$$

$$\Rightarrow \frac{5}{2y^2 - 7y + 3}\,dy = 1\,dx$$

$$\Rightarrow \int \frac{5}{2y^2 - 7y + 3}\,dy = \int 1\,dx$$

BY PARTIAL FRACTIONS

$$\frac{5}{2y^2 - 7y + 3} = \frac{5}{(2y-1)(y-3)}$$

$$\frac{5}{(2y-1)(y-3)} \equiv \frac{A}{2y-1} + \frac{B}{y-3}$$

$$5 \equiv A(y-3) + B(2y-1)$$

If $y = 3 \Rightarrow 5 = 5B \Rightarrow B = 1$

If $y = \frac{1}{2} \Rightarrow 5 = -\frac{5}{2}A \Rightarrow A = -2$

$$\Rightarrow \int \frac{1}{y-3} - \frac{2}{2y-1}\,dy = \int 1\,dx$$

$$\Rightarrow \ln|y-3| - \ln|2y-1| = x + C$$

$$\Rightarrow \ln\left|\frac{y-3}{2y-1}\right| = x + C$$

$$\Rightarrow \frac{y-3}{2y-1} = e^{x+C}$$

$$\Rightarrow \frac{y-3}{2y-1} = e^{x} \times e^{C}$$

$$\Rightarrow \frac{y-3}{2y-1} = Ae^{x} \quad (A = e^{C})$$

$$\Rightarrow y - 3 = (2y-1)Ae^{x}$$

$$\Rightarrow y - 3 = 2yAe^{x} - Ae^{x}$$

$$\Rightarrow Ae^{x} - 3 = 2yAe^{x} - y$$

$$\Rightarrow Ae^{x} - 3 = y(2Ae^{x} - 1)$$

$$\Rightarrow \frac{Ae^{x} - 3}{2Ae^{x} - 1} = y$$

$$\Rightarrow y = \frac{Ae^{x} - 3}{2Ae^{x} - 1}$$

Trapezium rule

Question 2 (**)

The values of y, for a curve with equation $y = f(x)$, have been tabulated below.

x	1	2.25	3.5	4.75	6
y	9	17	25	21	13

Use the trapezium rule with all the values from the above table to find an estimate for the integral

$$\int_1^6 f(x)\,dx.$$

$\boxed{}$, $\boxed{92.5}$

x	1	2.25	3.5	4.75	6
y	9	17	25	21	13

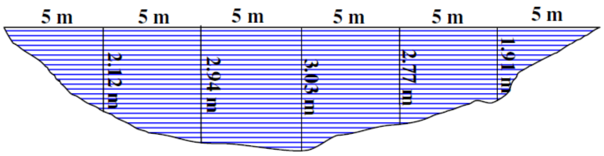
$$\int_1^6 f(x)\,dx \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$\approx \frac{1.25}{2}\left[9 + 13 + 2\times(17+25+21)\right]$$

$$\approx 92.5$$

Question 11 (***)

The figure below shows the cross section of a river.



The depth of the river, in metres, from one river bank directly across to the other river bank, is recorded at 5 metre intervals.

Estimate the cross sectional area of the river, by using the trapezium rule with all the measurements provided in the above figure.

$\boxed{}$, $\boxed{\approx 63.85 \text{ m}^2}$

$$\text{AREA} \approx \frac{\text{THICKNESS}}{2}\left[\text{FIRST} + \text{LAST} + 2\times\text{REST}\right]$$

$$\approx \frac{5}{2}\left[0 + 0 + 2(2.12 + 2.94 + 3.03 + 2.77 + 1.91)\right]$$

$$\approx 63.85$$

Exponentials and logs

Question 15 (***)

Solve the following logarithmic equation for x.

$$\log_a(x^2 - 10) - \log_a x = 2\log_a 3 .$$

$\boxed{x = 10,\ x \neq -1}$

$$\log_a(x^2-10) - \log_a x = 2\log_a 3$$

$$\Rightarrow \log_a\left(\frac{x^2-10}{x}\right) = \log_a 3^2$$

$$\Rightarrow \log_a\left(\frac{x^2-10}{x}\right) = \log_a 9$$

$$\Rightarrow \frac{x^2-10}{x} = 9$$

$$\Rightarrow x^2-10 = 9x$$

$$\Rightarrow x^2-9x-10 = 0$$

$$\Rightarrow (x+1)(x-10) = 0$$

$$\Rightarrow x = \cancel{-1},\ 10$$

Question 29 (***+)

Solve the following simultaneous logarithmic equations

$$\log_2\left(xy^2\right) = 0$$

$$\log_2\left(x^2y\right) = 3.$$

$\boxed{}$, $\boxed{x = 4,\ y = \frac{1}{2}}$

$$\begin{cases} \log_2(xy^2) = 0 \\ \log_2(x^2y) = 3 \end{cases}$$

$$\begin{cases} \log_2 x + \log_2 y^2 = 0 \\ \log_2 x^2 + \log_2 y = 3 \end{cases}$$

$$\begin{cases} \log_2 x + 2\log_2 y = 0 \\ 2\log_2 x + \log_3 y = 3 \end{cases}$$

$$\begin{cases} X + 2Y = 0 \\ 2X + Y = 3 \end{cases}$$

$$-2X - 4Y = 0$$
$$2X + Y = 3$$
$$-3Y = 3$$

$$\boxed{\begin{matrix} Y = -1 \\ X = 2 \end{matrix}}$$

- $\log_2 y = -1 \Rightarrow y = \frac{1}{2}$
- $\log_2 x = 2 \Rightarrow x = 4$

Alternative

$$\begin{cases} \log_2(xy^2) = 0\log_2 2 \\ \log_2(x^2y) = 3\log_2 2 \end{cases}$$

$$\begin{cases} \log_2(xy^2) = \log_2 2^0 \\ \log(x^2y) = \log_2 2^3 \end{cases}$$

$$\begin{cases} xy^2 = 1 \\ x^2y = 8 \end{cases}$$

$$\begin{cases} x^2y^4 = 1 \\ x^2y = 8 \end{cases}$$

divide

$$\frac{x^2y^4}{x^2y} = \frac{1}{8}$$

$$y^3 = \frac{1}{8}$$

$$y = \frac{1}{2}$$

$$x\left(\frac{1}{2}\right)^2 = 1$$

$$\frac{1}{4}x = 1$$

$$x = 4$$

Question 38 (***+)

The population P of a certain town in time t years is modelled by the equation

$$P = A \times 10^{kt}, \quad t \geq 0,$$

where A and k are non zero constants.

When $t = 3$, $P = 19000$ and when $t = 6$, $P = 38000$.

Find the value of A and the value of k, correct to 2 significant figures.

$\boxed{\quad}$, $\boxed{A = 9500,\ k = 0.10}$

$P = A \times 10^{kt}$

$$\left.\begin{array}{l} 19000 = A \times 10^{3k} \\ 38000 = A \times 10^{6k} \end{array}\right\} \Rightarrow \frac{\cancel{A} \times 10^{6k}}{\cancel{A} \times 10^{3k}} = \frac{38000}{19000}$$

$$10^{3k} = 2$$

$$\log 10^{3k} = \log 2$$

$$3k \log 10 = \log 2$$

$$3k = \log 2$$

$$k = \tfrac{1}{3}\log 2 \approx 0.10$$

Hence $A \times 10^{3k} = 19000$

$$2A = 19000$$

$$A = 9500$$

Question 45 (****)

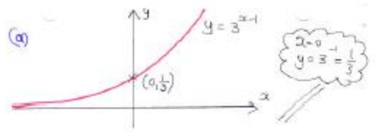
$$y = 3^{x-1}, \quad x \in \mathbb{R}.$$

a) Sketch the graph of $y = 3^{x-1}$ showing the coordinates of all intercepts with the coordinate axes.

b) Find to 3 significant figures the x coordinate of the point where the curve $y = 3^{x-1}$ intersects with the straight line with equation $y = 10$.

c) Determine to 3 significant figures the x coordinate of the point where the curve $y = 3^{x-1}$ intersects with the curve $y = 2^x$.

$\boxed{3.10}$, $\boxed{2.71}$



(b) $3^{x-1} = 10$

$\Rightarrow \log_{10} 3^{x-1} = \log_{10} 10$

$\Rightarrow (x-1)\log_{10} 3 = 1$

$\Rightarrow x-1 = \frac{1}{\log_{10} 3}$

$\Rightarrow x = 1 + \frac{1}{\log_{10} 3}$

$\Rightarrow x \approx 3.10$ (3 s.f.)

(c) $3^{x-1} = 2^x$

$\Rightarrow \log_{10}(3^{x-1}) = \log_{10} 2^x$

$\Rightarrow (x-1)\log_{10} 3 = x\log_{10} 2$

$\Rightarrow x\log_{10} 3 - \log_{10} 3 = x\log_{10} 2$

$\Rightarrow x\log_{10} 3 - x\log_{10} 2 = \log_{10} 3$

$\Rightarrow x[\log_{10} 3 - \log_{10} 2] = \log_{10} 3$

$\Rightarrow x = \frac{\log_{10} 3}{\log_{10} 3 - \log_{10} 2}$

$\Rightarrow x \approx 2.71$ (3 s.f.)

Question 47 (****)

In 1970 the average weekly pay of footballers in a certain club was £100.

The average weekly pay, £P, is modelled by the equation

$$P = A \times b^t,$$

where t is the number of years since 1970, and A and b are positive constants.

In 1991 the average weekly pay of footballers in the same club had risen to £740.

a) Find the value of A and show that $b = 1.10$, correct to three significant figures.

b) Determine the year when the average weekly pay of footballers in this club will first exceed £10000.

$\boxed{}$, $\boxed{A = 100}$, $\boxed{2019}$

(a) $\boxed{P = A \times b^t}$

- $t=0$ $P=100$ $\Rightarrow 100 = A \times b^0$ $\boxed{100 = A}$
- $t=21$ $P=740$ $\Rightarrow 740 = 100 \times b^{21}$

$7.4 = b^{21}$

$\sqrt[21]{7.4} \approx b$

$b \approx 1.099998...$

$b \approx 1.10$ (3 s.f.)

AS REQUIRED

(b) $\boxed{P = 100 \times 1.1^t}$

$\Rightarrow 10000 = 100 \times 1.1^t$

$\Rightarrow 100 = 1.1^t$

$\Rightarrow \log 100 = \log(1.1)^t$

$\Rightarrow 2 = t\log(1.1)$

$\Rightarrow t = \frac{2}{\log(1.1)}$

$\Rightarrow t \approx 48.32...$

$t = 49$

$\therefore 1970 + 49 = 2019$

Question 6 (**)

A curve has equation

$$y = A\mathrm{e}^{kx},$$

where A and k are non zero constants.

The curve passes through the points $(0,4)$ and $(12,16)$.

a) Find the value of A and the exact value of k.

b) Determine the value of y when $x = 30$

$\boxed{A = 4}$, $\boxed{k = \frac{1}{6}\ln 2}$, $\boxed{y = 128}$

$\boxed{y = Ae^{kx}}$ $(0,4)$ & $(12,16)$

(a) • $4 = Ae^{0k}$

$4 = Ae^{0}$

$A = 4$

• $\boxed{y = 4e^{kx}}$

$\Rightarrow 16 = 4e^{12k}$

$\Rightarrow 4 = e^{12k}$

$\Rightarrow \ln 4 = 12k$

$\Rightarrow k = \frac{1}{12}\ln 4$

$\Rightarrow k = \frac{1}{6}\ln 2$

(b) $\boxed{y = 4e^{(\frac{1}{6}\ln 2)x}}$

when $x = 30$

$\Rightarrow y = 4e^{5\ln 2}$

$\Rightarrow y = 4e^{\ln 32}$

$\Rightarrow y = 4 \times 32$

$\Rightarrow y = 128$

Question 7 (**)

A cup of coffee is cooling down in a room.

The temperature T °C of the coffee, t minutes after it was made is modelled by

$$T = 20 + 50\mathrm{e}^{-\frac{t}{15}}, \quad t > 0.$$

a) State the temperature of the coffee when it was first made.

b) Find the temperature of the coffee, after 30 minutes.

c) Calculate, to the nearest minute, the value of t when the temperature of the coffee has reached 35°C.

$\boxed{}$, $\boxed{T = 70}$, $\boxed{26.8\text{°C}}$, $\boxed{t = 18}$

(a) $T = 20 + 50e^{-\frac{1}{15}t}$

• when $t=0$

$T = 20 + 50e^{0}$

$T = 20 + 50$

$T = 70°C$

(b) • when $t=30$

$T = 20 + 50e^{-2}$

$T \approx 26.8°C$

(c) when $T = 35$

$35 = 20 + 50e^{-\frac{1}{15}t}$

$\Rightarrow 15 = 50e^{-\frac{1}{15}t}$

$\Rightarrow \frac{3}{10} = e^{-\frac{1}{15}t}$

$\Rightarrow \frac{10}{3} = e^{\frac{1}{15}t}$

$\Rightarrow \ln\left(\frac{10}{3}\right) = \frac{1}{15}t$

$\Rightarrow t = 15\ln\left(\frac{10}{3}\right)$

$\Rightarrow t \simeq 18$

Question 9 (**+)

A microbiologist models the population of bacteria in culture by the equation

$$P = 1000 - 950e^{-\frac{1}{2}t}, \; t > 0$$

where P is the number of bacteria in time t hours.

a) Find the initial number of bacteria in the culture.

b) Show mathematically that the limiting value for P is 1000.

c) Find the value of t when $P = 500$.

$\boxed{P_0 = 50}$, $\boxed{t \approx 1.28}$

(a) $P = 1000 - 950e^{-\frac{1}{2}t}$

• $t=0$ $\quad P = 1000 - 950e^{0}$

$P = 50$

(b) As $t \to \infty$ $\quad e^{-\frac{1}{2}t} \to 0$

$950e^{-\frac{1}{2}t} \to 0$

$P \to 1000$

(c) • $P = 500$

$500 = 1000 - 950e^{-\frac{1}{2}t}$

$\Rightarrow 950e^{-\frac{1}{2}t} = 500$

$\Rightarrow e^{-\frac{1}{2}t} = \frac{10}{19}$

$\Rightarrow e^{\frac{1}{2}t} = \frac{19}{10}$

$\Rightarrow \frac{1}{2}t = \ln\left(\frac{19}{10}\right)$

$\Rightarrow t = 2\ln\left(\frac{19}{10}\right) \approx 1.28$

Question 16 (***)

A car tyre develops a puncture.

The tyre pressure P, measured in suitable units known as p.s.i., t minutes after the tyre got punctured is given by the expression

$$P = 8 + 32e^{-kt}, \; t > 0,$$

where k is a positive constant.

a) State the tyre pressure when the tyre got punctured.

The tyre pressure halves 2 minutes after the puncture occurred.

b) Show that $k = 0.4904$, correct to 4 significant figures.

c) Calculate the time it takes for the tyre pressure to drop to 12 p.s.i.

d) Find the rate at which the pressure of the tyre is changing one minute after the puncture occurred.

$\boxed{}$, $\boxed{P = 40}$, $\boxed{t \approx 4.24}$, $\boxed{-9.61 \text{ p.s.i/min}}$

(a) $P = 8 + 32e^{-kt}$

when $t=0$

$P = 8 + 32e^{0}$

$P = 40$

(b) $20 = 8 + 32e^{-2k}$

$\Rightarrow 12 = 32e^{-2k}$

$\Rightarrow \frac{3}{8} = e^{-2k}$

$\Rightarrow \frac{8}{3} = e^{2k}$

$\Rightarrow 2k = \ln\frac{8}{3}$

$\Rightarrow k = \frac{1}{2}\ln\frac{8}{3} \approx 0.4904$

(c) $P = 8 + 32e^{-0.4904t}$

$\Rightarrow 12 = 8 + 32e^{-0.4904t}$

$\Rightarrow 4 = 32e^{-0.4904t}$

$\Rightarrow \frac{1}{8} = e^{-0.4904t}$

$\Rightarrow 8 = e^{0.4904t}$

$\Rightarrow 0.4904t = \ln 8$

$\Rightarrow t \approx 4.24$

(d) $P = 8 + 32e^{-0.4904t}$

$\frac{dP}{dt} = 32(-0.4904)e^{-0.4904t}$

$\frac{dP}{dt} = -15.6929e^{-0.4904t}$

$\left.\frac{dP}{dt}\right|_{t=1} = -9.61$

Question 17 (***)

The population P, in thousands, of a colony of rabbits in time t years after a certain instant, is given by

$$P = 5 + a\mathrm{e}^{-bt}, \ t \geq 0$$

where a and b are positive constants.

It is given that the initial population is 8 thousands rabbits, and one year later this population has reduced by 2 thousands.

a) Find the value of a and the value of b.

b) Explain mathematically, why the population can never reach 1000, according to this model.

$\boxed{a = 3}$, $\boxed{b = \ln 3}$

(a) $P = 5 + a\mathrm{e}^{-bt}$

• $t=0 \quad P=8 \Rightarrow 8 = 5 + a\mathrm{e}^{0}$
$8 = 5 + a$
$a = 3$

• $t=1 \quad P=6 \Rightarrow 6 = 5 + 3\mathrm{e}^{-b \times 1}$
$\Rightarrow 1 = 3\mathrm{e}^{-b}$
$\Rightarrow \frac{1}{3} = \mathrm{e}^{-b}$
$\Rightarrow 3 = \mathrm{e}^{b}$
$\Rightarrow b = \ln 3$

(b) As $t \to \infty$
$\mathrm{e}^{-bt} \to 0$
$\therefore P \to 5$

IE POPULATION CAN ONLY REDUCE TO 5000

∴ CAN NEVER DROP TO 1000

Question 24 (***+)

The volume of water in a tank V m^3, t hours after midnight, is given by the equation

$$V = 10 + 8\mathrm{e}^{-\frac{1}{12}t}, \ t > 0.$$

a) State the volume of water in the tank at midnight.

b) Find the time, using 24 hour clock notation, when the volume of the water in the tank is 14 m^3.

c) Determine the rate at which the volume of the water is changing at midday, explaining the significance of its sign.

d) State the limiting value of V.

$\boxed{}$, $\boxed{V = 18}$, $\boxed{08:19}$, $\boxed{-\frac{2}{3\mathrm{e}} \approx -0.245, \text{ decrease}}$, $\boxed{t \to \infty, \ V \to 10}$

(a) $V = 10 + 8e^{-\frac{1}{12}t}$

$t=0 \quad V = 10 + 8e^{0} = 18$

(b) $14 = 10 + 8e^{-\frac{1}{12}t}$

$4 = 8e^{-\frac{1}{12}t}$

$\frac{1}{2} = e^{-\frac{1}{12}t}$

$2 = e^{\frac{1}{12}t}$

$\ln 2 = \frac{1}{12}t$

$t = 12\ln 2$

$t \simeq 8.317\ldots$

$\therefore 0.317 \times 60 \approx 19$

$\therefore 08:19$

(c) $V = 10 + 8e^{-\frac{1}{12}t}$

$\frac{dV}{dt} = -\frac{2}{3}e^{-\frac{1}{12}t}$

$\left.\frac{dV}{dt}\right|_{t=12} = -\frac{2}{3}e^{-1} \approx -0.245$

(minus = decrease)

(d) As $t \to \infty \quad e^{-\frac{1}{12}t} \to 0$

$8e^{-\frac{1}{12}t} \to 0$

$V \to 10$

Question 44 (****)

Find, in exact simplified form, the solution of each of the following equations.

a) $e^{2x-3} = 2e$.

b) $\ln(2y-1) = 1 + \ln(e - y)$.

$\boxed{x = \frac{1}{2}(4 + \ln 2)}$, $\boxed{y = \frac{e^2+1}{e+2}}$

(a) $e^{2x-3} = 2e$

$\Rightarrow \frac{e^{2x-3}}{e} = 2$

$\Rightarrow e^{2x-4} = 2$

$\Rightarrow 2x-4 = \ln 2$

$\Rightarrow 2x = 4 + \ln 2$

$\Rightarrow x = \frac{1}{2}(4 + \ln 2)$

(b) $\ln(2y-1) = 1 + \ln(e-y)$

$\Rightarrow \ln(2y-1) - \ln(e-y) = 1$

$\Rightarrow \ln\left(\frac{2y-1}{e-y}\right) = 1$

$\Rightarrow \frac{2y-1}{e-y} = e^{1}$

$\Rightarrow 2y-1 = e^2 - ey$

$\Rightarrow 2y + ey = e^2 + 1$

$\Rightarrow y(2+e) = e^2 + 1$

$\Rightarrow y = \frac{e^2+1}{e+2}$

7 It is thought that the global population of tigers is falling exponentially. Estimates suggest that in 1970 there were 37 000 tigers but by 1980 the number had dropped to 22 000.

a A model of the form $T = ka^n$ is suggested, connecting the number of tigers (T) with the number of years (n) after 1970.

i Show that $22\,000 = ka^{10}$.

ii Write another similar equation and solve them to find k and a.

b What does the model predict the tiger population will be in 2020?

c When the population reaches 1000, the tiger population will be described as 'near extinction'. In which year will this happen?

8 A zoologist believes that the population of fish in a small lake is growing exponentially. He collects data about the number of fish every 10 days for 50 days. The data are given in this table:

Time (days)	0	10	20	30	40	50
Number of fish	35	42	46	51	62	71

The zoologist proposes a model of the form $N = Ae^{kt}$ where N is the number of fish and t is time in days. In order to estimate the values of the constant A and k he plots a graph with t on the horizontal axis and $\ln N$ on the vertical axis.

a Explain why, assuming the zoologist's model is correct, this graph will be approximately a straight line.

b Complete the table of values for the graph:

t	0	10	20	30	40	50
$\ln N$	3.56	3.74	3.83	3.93		4.26

c Find the equation of the line of best fit for this table. (Do not draw the graph.) Hence estimate the values of A and k.

d Use this model to predict the number of fish in the lake when $t = 260$.

e The zoologist finds that the number of fish in the lake after 260 days is actually 720. Suggest one reason why the observed data does not fit the prediction.

7 a i Proof

ii $k = 37\,000$, $a = 0.949$

b 2700

c 2039

8 a $\ln(N) = kt + \ln(A)$

b 4.13, 4.26

c $\ln(N) = 0.0137t + 3.56$; $N = 35.2e^{0.0137t}$

d 1240

e Size of the lake limits indefinite growth; seasonal variation

c

4 a

b

c

5 27

6 a

7

Series – Arithmetic, Geometric

Question 20 (***) non calculator

The sum of the first 20 terms of an arithmetic series is 1070.

The sum of its fifth term and its tenth term is 65.

a) Find the first term and the common difference of the series.

b) Calculate the sum of the first 30 terms of the series.

$\boxed{a = -13}$, $\boxed{d = 7}$, $\boxed{2655}$

(a) $\boxed{S_n = \frac{n}{2}\left[2a + (n-1)d\right]}$

$\Rightarrow 1070 = \frac{20}{2}\left[2a + 19d\right]$

$\Rightarrow 1070 = 10\left[2a + 19d\right]$

$\Rightarrow \boxed{107 = 2a + 19d}$

$u_5 + u_{10} = 65$

$(a + 4d) + (a + 9d) = 65$

$\boxed{2a + 13d = 65}$

$2a = 107 - 19d$

$2a = 65 - 13d$

$107 - 19d = 65 - 13d$

$42 = 6d$

$d = 7$

& $2a = 65 - 13d$

$2a = 65 - 13 \times 7$

$2a = 65 - 91$

$2a = -26$

$a = -13$

(b) $S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow S_{30} = \frac{30}{2}\left[2 \times (-13) + 29 \times 7\right]$

$\Rightarrow S_{30} = 15\left[-26 + 203\right]$

$\Rightarrow S_{30} = 15 \times 177$

$\Rightarrow S_{30} = 2655$

1770
885
2655

Question 43 (***+)

The n^{th} term of an arithmetic series is denoted by u_n.

a) Given that $u_7 = 7u_{19}$ and $u_{31} = 25$, show that the common difference of the series is 2.5.

The last term of the series is 150.

b) Determine the number of terms in the series.

c) Find the sum of the last 20 terms of the series.

$\boxed{81}$, $\boxed{2525}$

$\boxed{u_n = a+(n-1)d}$

(a) • $u_7 = 7u_{19}$

$\Rightarrow a+6d = 7(a+18d)$

$\Rightarrow a+6d = 7a+126d$

$\Rightarrow -120d = 6a$

$\boxed{-20d = a}$

• $u_{31} = 25$

$a+30d = 25$

$\boxed{a = 25-30d}$

$-20d = 25-30d$

$10d = 25$

$d = 2.5$ // AS REQUIRED

(b) $a = -20d$

$a = -20(2.5)$

$\boxed{a = -50}$

• $u_n = a+(n-1)d$

$\Rightarrow 150 = -50+(n-1)\times 2.5$

$\Rightarrow 200 = 2.5(n-1)$

$\Rightarrow \frac{200}{2.5} = n-1$

$\Rightarrow \frac{800}{10} = n-1$

$\Rightarrow n-1 = 80$

$\therefore n = 81$

(c) $\underbrace{150 + 147.5 + 145 + \dots}_{20 \text{ terms}}$

$a = 150$

$d = -2.5$

$n = 20$

$S_n = \frac{n}{2}\left[2a+(n-1)d\right]$

$S_{20} = \frac{20}{2}\left[2\times150 + 19\times(-2.5)\right]$

$S_{20} = 10\left[300 - 19\times 2.5\right]$

$S_{20} = 3000 - 19\times 25$

$S_{20} = 3000 - 475$

$S_{20} = 2525$ //

Question 68 (***+)

The first three terms of an arithmetic series are

$$8-k, \quad 2k+1 \text{ and } 4k-1 \text{ respectively,}$$

where k is a constant.

a) Show clearly that $k=5$.

b) Find the sum of the first fifteen terms of the series.

c) Determine how many terms of the series have a value less than 400.

$\boxed{S_{15}=885}$, $\boxed{50 \text{ terms}}$

(a) ARITHMETIC $\Rightarrow$ $u_3-u_2=u_2-u_1$

$(4k-1)-(2k+1)=(2k+1)-(8-k)$

$2k-2=3k-7$

$5=k$

(b) $u_1=8-k=3$

$u_2=2k+1=11$ $\big)+8$

$u_3=4k-1=19$ $\big)+8$

$\therefore a=3$, $d=8$, $n=15$

$S_n=\frac{n}{2}\left[2a+(n-1)d\right]$

$S_{15}=\frac{15}{2}\left[2\times3+14\times8\right]$

$S_{15}=\frac{15}{2}\times118$

$S_{15}=885$

(c) $u_k=a+(n-1)d$

$400=3+(n-1)\times8$

$400=3+8n-8$

$405=8n$

$n=50.625$

$\therefore n=50$

i.e. 50 TERMS

Question 72 (***+)

The n^{th} term of an arithmetic progression is denoted by u_n, and given by

$$u_n=2n+7.$$

Determine the value of N given that $\displaystyle\sum_{n=1}^{N} u_n = 2100$.

$\boxed{42}$

$u_n = 2n+7$ ie $9, 11, 13, 15, 17, \dots$

$$\sum_{n=1}^{N} u_n = 2100$$

$\Rightarrow u_1 + u_2 + u_3 + \dots + u_N = 2100$

$\Rightarrow 9 + 11 + 13 + \dots + u_N = 2100$

THIS IS AN A.P, $a = 9$

$d = 2$

$S_n = 2100$

$n = 30 \times 38 = 1240$
$n = 40 \times 48 = 1920$
$n = 41 \times 49 \neq 2100$
$n = 42 \times 50 = 2100$

$\Rightarrow S_n = \frac{n}{2}\left[2a + (n-1)d\right]$

$\Rightarrow 2100 = \frac{n}{2}\left[2 \times 9 + (n-1) \times 2\right]$

$\Rightarrow 2100 = \frac{n}{2}\left[18 + 2n - 2\right]$

$\Rightarrow 2100 = \frac{n}{2}(2n + 16)$

$2100 = n(n+8)$

$\therefore n = 42$

Question 5 (***)

A novelist is planning to write a new book.

He plans to write 15 pages in the first week, 17 pages in the second week, 19 pages in the third week, and so on, so that he writes an extra two pages each week compared with the previous week.

a) Find the number of pages he plans to write in the tenth week.

b) Determine how many pages he plans to write in the first ten weeks.

The novelist sticks to his plan and produces a book with 480 pages, after n weeks.

c) Use algebra to determine the value of n.

$\boxed{33}$, $\boxed{240}$, $\boxed{n = 16}$

(a) $\boxed{u_n = a+(n-1)d}$

$u_{10} = 15+9\times 2$

$u_{10} = 15+18$

$u_{10} = 33$

(b) $\boxed{S_n = \frac{n}{2}\left[2a+(n-1)d\right]}$

$\Rightarrow S_{10} = \frac{10}{2}\left[2\times 15+9\times 2\right]$

$\Rightarrow S_{10} = 5\left[30+18\right]$

$\Rightarrow S_{10} = 240$

(c) $\boxed{S_n = \frac{n}{2}\left[2a+(n-1)d\right]}$

$480 = \frac{n}{2}\left[30+(n-1)\times 2\right]$

$480 = \frac{n}{2}\left[30+2n-2\right]$

$480 = \frac{n}{2}\left[2n+28\right]$

$480 = n(n+14)$

SINCE SOLUTION IS A POSITIVE INTEGER

IF $n=15$ $\quad 15\times(29) = \ldots\ 435$

$n=16$ $\quad 16\times 30 = \ldots\ 480$

$\therefore n=16$

Question 16 (***+)

A length of rope is wrapped neatly around a circular pulley.

The length of the rope in the first coil (the nearest to the pulley) is 60 cm, and each successive coil of rope (outwards) is 3.5 cm longer than the previous one.

The outer coil has a length of 144 cm.

Show that total length of the rope is 25.5 metres.

proof

$a = 60$

$d = 3.5$

$u_n = L = 144$

• $u_n = a+(n-1)d$

$\Rightarrow 144 = 60+(n-1)\times 3.5$

$\Rightarrow 84 = (n-1)\times\frac{7}{2}$

$\Rightarrow 12 = \frac{1}{2}(n-1)$

$\Rightarrow 24 = n-1$

$\Rightarrow n = 25$

• $S_n = \frac{n}{2}\left[a+L\right]$

$\Rightarrow S_{25} = \frac{25}{2}\left[60+144\right]$

$\Rightarrow S_{25} = \frac{25}{2}\times 204$

$\Rightarrow S_{25} = 25\times 102$

$\Rightarrow S_{25} = 2500+50$

$\therefore 2550$ cm

or

25.5 m

AS REQUIRED

Geometric

Question 16 (**+)

The third and the sixth term of a geometric series is 4 and 6.912, respectively.

a) Find the exact value of the first term and the common ratio of the series.

b) Calculate, to three significant figures, the sum of the first ten terms of the series.

$\boxed{}$, $\boxed{a = \frac{25}{9}}$, $\boxed{r = \frac{6}{5}}$, $\boxed{S_{10} \approx 72.1}$

(a) $\boxed{u_n = ar^{n-1}}$

$$\left.\begin{array}{l} u_3 = 4 \\ u_6 = 6.912 \end{array}\right\} \Rightarrow \boxed{\begin{array}{l} 4 = ar^2 \\ 6.912 = ar^5 \end{array}}$$

DIVIDE EQUATIONS

$$\frac{\cancel{a}r^5}{\cancel{a}r^2} = \frac{6.912}{4} \Rightarrow r^3 = 1.728$$

$$\Rightarrow r = 1.2$$

HENCE $4 = a \times 1.2^2$

$4 = 1.44 \times a$

$a = 25/9$

(b) $\boxed{S_n = \frac{a(1-r^n)}{1-r}}$

$$\Rightarrow S_{10} = \frac{\frac{25}{9}(1-1.2^{10})}{1-1.2}$$

$$\Rightarrow S_{10} = 72.10745031\ldots$$

$$\Rightarrow S_{10} = 72.1$$

(3 sf)

Question 26 (***)

The sum to infinity of a geometric progression of positive terms is 270 and the sum of its first two terms is 240.

Find the first term and the common ratio of the progression.

$\boxed{a = 180}$, $\boxed{r = \frac{1}{3}}$

$S_\infty = 270$

$\frac{a}{1-r} = 270$

$\boxed{a = 270(1-r)}$

$S_2 = 240$

$a + ar = 240$

$\boxed{a(1+r) = 240}$

$270(1-r)(1+r) = 240$

$(1-r)(1+r) = \frac{8}{9}$

$1 - r^2 = \frac{8}{9}$

$\frac{1}{9} = r^2$

$r = +\frac{1}{3}$ (POSITIVE TERMS)

& $a = 270(1-r)$

$a = 270\left(1-\frac{1}{3}\right)$

$a = 180$

Question 36 (***+)

The first three terms of a geometric series are given below as functions of x.

$$x^2, \quad (x+12) \quad \text{and} \quad (2x-3).$$

a) Show that x is a solution of the equation

$$x^3 - 2x^2 - 12x - 72 = 0.$$

b) Show clearly that $x = 6$ is the only solution of the above equation.

c) Find the sum to infinity of the series.

$\boxed{}$, $\boxed{S_\infty = 72}$

(a) $u_1 = x^2$
$u_2 = x+12$
$u_3 = 2x-3$

IF GEOMETRIC $\frac{u_2}{u_1} = \frac{u_3}{u_2}$

$$\Rightarrow \frac{x+12}{x^2} = \frac{2x-3}{x+12}$$
$$\Rightarrow (x+12)^2 = (2x-3)x^2$$
$$\Rightarrow x^2+24x+144 = 2x^3-3x^2$$
$$\Rightarrow 0 = 2x^3-4x^2-24x+144$$
$$\Rightarrow x^3-2x^2-12x+72=0$$

AS REQUIRED

(b) $(x-6)\ (x^2+Ax+12)$ BY INSPECTION

$-6Ax$

$12x$

LOOKING AT COEFFICIENT OF x

$$12x - 6Ax = -12x$$
$$12 - 6A = -12$$
$$24 = 6A$$
$$A = 4$$

$\therefore (x-6)(x^2+4x+12)=0$

$b^2-4ac = 4^2-4\times1\times12 = 16-48 = -32 < 0$

NO MORE SOLUTIONS, ONLY SOLUTION $x=6$

(c) $u_1 = x^2 = 36$
$u_2 = x+12 = 18$
$u_3 = 2x-3 = 9$

$\therefore$ $a = 36$, $r = \frac{1}{2}$

HENCE $S_\infty = \frac{a}{1-r}$

$S_\infty = \frac{36}{1-\frac{1}{2}}$

$S_\infty = 72$

Question 44 (***+)

The second and third term of a geometric progression are 9.6 and 9.216, respectively.

a) Show that the sum to infinity of the progression is 250.

The sum of the first k terms of the progression is greater than 249.

b) Show clearly that

$$0.96^k < 0.004.$$

c) Hence determine the smallest value of k.

$\boxed{k = 136}$

(a) $r = \frac{u_3}{u_2} = \frac{9.216}{9.6} = \frac{24}{25} = 0.96$

$\therefore u_n = ar^{n-1}$

$9.6 = a \times 0.96^1$

$\boxed{a = 10}$

$\therefore S_\infty = \frac{a}{1-r} = \frac{10}{1-0.96} = 250$ // AS REQUIRED

(b) $S_n = \frac{a(1-r^n)}{1-r}$

$\Rightarrow \frac{10(1-0.96^k)}{1-0.96} > 249$

$\Rightarrow \frac{10(1-0.96^k)}{0.04} > 249$

$\Rightarrow 1 - 0.96^k > 0.996$

$\Rightarrow -0.96^k > -0.004$

$\Rightarrow 0.96^k < 0.004$ // AS REQUIRED

(c) BY LOGS OR TRIAL & IMPROVEMENT

$\Rightarrow 0.96^k < 0.004$

$\Rightarrow \log(0.96^k) < \log(0.004)$

$\Rightarrow k\log(0.96) < \log 0.004$

$\Rightarrow k > \frac{\log(0.004)}{\log(0.96)}$

$\log(0.96)$ IS NEGATIVE

$\Rightarrow k > 135.257$

$\therefore k = 136$

Question 4 (***+)

The manufacturer of a certain brand of washing machine is to replace an old model with a new model. There will be a "phase out" period for the old model and a "phase in" period for the new model, both lasting 24 months and starting at the same time.

On the first month of the phase out period 5000 old washing machines will be produced and each month thereafter, this figure will reduce by 20%.

a) Show that on the fifth month of the "phase out" period 2048 old washing machines will be produced.

b) Find how many old washing machines will be produced during the "phase out" period.

On the first month of the "phase in" period 1000 new washing machines will be produced and each month thereafter, this figure will increase by 5%.

c) Calculate how many new washing machines will be produced on the last month of the "phase in" period.

On the k^{th} month of the "phase in/phase out" period, for the first time more new washing machines will be produced than old washing machines.

d) Show that k satisfies

$$\left(\frac{21}{16}\right)^{k-1} > 5.$$

e) Use logarithms to determine the value of k.

$\boxed{24881 \text{ or } 24882}$, $\boxed{3071 \text{ or } 3072}$, $\boxed{k = 7}$

a) $a = 5000$, $r = 0.8$, $n = 5$

$u_n = ar^{n-1}$

$u_5 = 5000 \times 0.8^4$

$u_5 = 2048$

b) $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{24} = \dfrac{5000(1-0.8^{24})}{1-0.8}$

$S_{24} \simeq 24881.94\ldots$

$\therefore$ APPROX 24882 (OR 24881)

c) $a = 1000$, $r = 1.05$, $n = 24$

$u_n = ar^{n-1}$

$u_{24} = 1000 \times 1.05^{23}$

$u_{24} \simeq 3071.52\ldots$

$\therefore$ APPROX 3072 (OR 3071)

d) OLD $u_n = 5000 \times 0.8^{n-1}$

NEW $u_n = 1000 \times 1.05^{n-1}$

• WHEN $n = k$

$\Longrightarrow 1000 \times 1.05^{k-1} > 5000 \times 0.8^{k-1}$

$\Longrightarrow 1.05^{k-1} > 5 \times 0.8^{k-1}$

$\Longrightarrow \dfrac{1.05^{k-1}}{0.8^{k-1}} > 5$

$\Longrightarrow \left(\dfrac{1.05}{0.8}\right)^{k-1} > 5$

$\Longrightarrow \left(\dfrac{21}{16}\right)^{k-1} > 5$ AS REQUIRED

e) TAKING LOGS BASE 10

$\log\left(\frac{21}{16}\right)^{k-1} > \log 5$

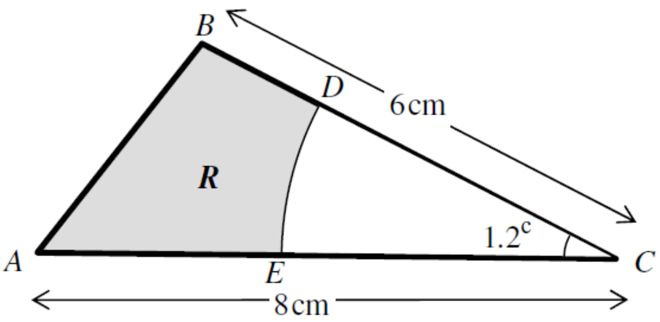
$(k-1)\log\left(\frac{21}{16}\right) > \log 5$

$k-1 > 5.918\ldots$

$k > 6.918\ldots$ $\quad \therefore k = 7$

Arc length, Sector area

Question 15 (**+)



The figure above shows a triangle ABC where the lengths of AC and BC are 8 cm and 6 cm, respectively. The angle BCA is 1.2 radians.

a) Find the length of AB.

b) Determine the area of the triangle ABC.

A circular arc with centre at C and radius 4 cm is drawn inside the triangle.

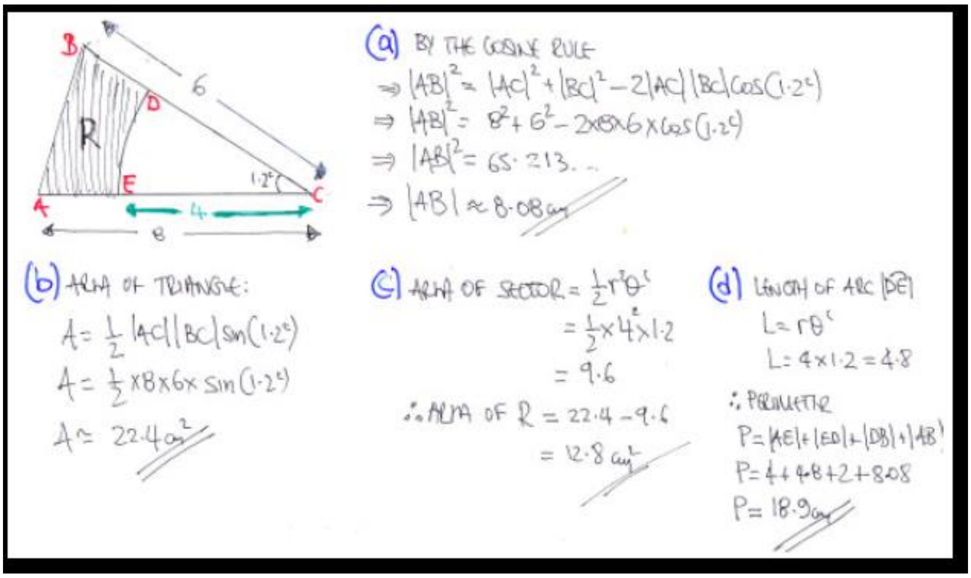
The arc intersects the triangle at the points D and E.

The shaded region R is bounded by the straight lines EA, AB, BD and the arc ED.

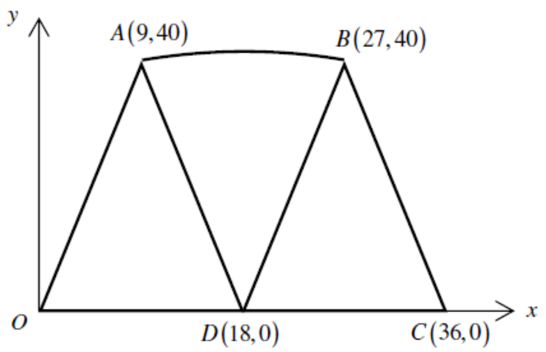
c) Calculate the area of R.

d) Calculate the perimeter of R.

$\boxed{}$, $\boxed{|AB| \approx 8.08}$, $\boxed{\text{area}_{ABC} \approx 22.4}$, $\boxed{\text{area}_R \approx 12.8}$, $\boxed{\text{perimeter}_R \approx 18.9}$



Question 21 (***)



The figure above shows the cross section of a river dam modelled in a system of coordinate axes where all units are in metres.

The cross section of the dam consists of a circular sector ADB and two isosceles triangles OAD and DBC.

The cross section of the dam consists of a circular sector ADB and two isosceles triangles OAD and DBC.

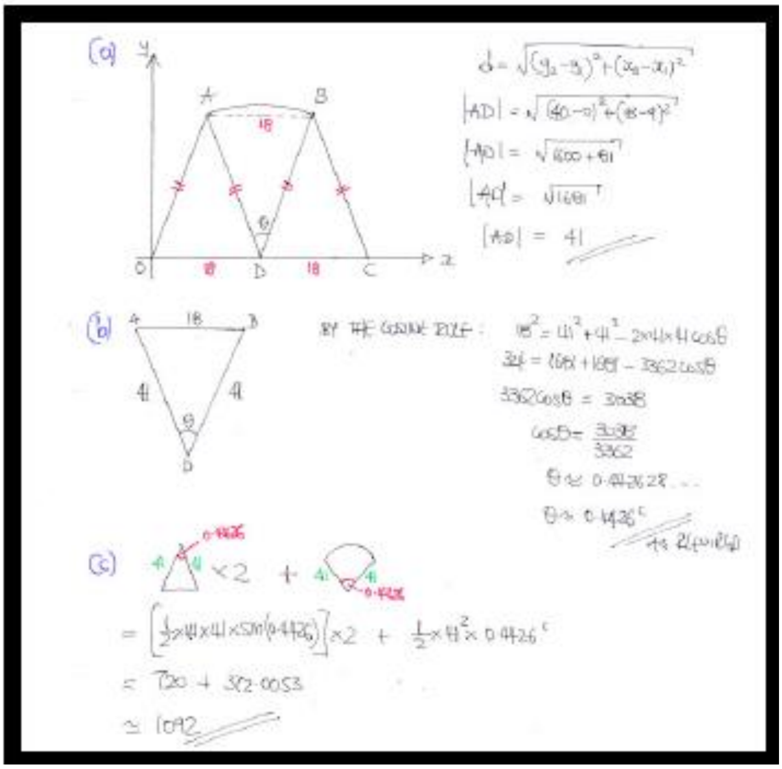
The coordinates of the points A, B, C and D are $(9,40)$, $(27,40)$, $(36,0)$ and $(18,0)$, respectively.

a) Find the length of AD.

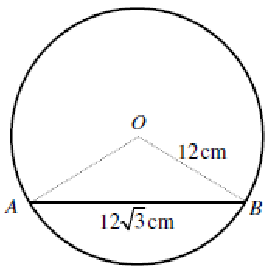
b) Show that the angle ADB is approximately 0.4426 radians.

c) Hence determine to the nearest m^2 the cross sectional area of the dam.

$\boxed{|AD| = 41}$, $\boxed{\text{area} \approx 1092}$



Question 35 (***+)



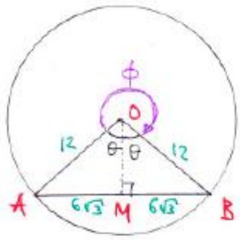
The figure above shows a circle with centre at O and radius 12 cm.

The chord AB has a length of $12\sqrt{3}$ cm.

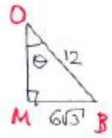
a) Show that the angle AOB is $\frac{2\pi}{3}$ radians.

b) Find, in exact form, the area of the **major** segment bounded by the chord AB.

$\boxed{\text{area} = 12\left(3\sqrt{3}+8\pi\right)}$



(a)



$\sin\theta = \frac{6\sqrt{3}}{12} = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}$

$\therefore \ A\hat{O}B = 2 \times \frac{\pi}{3} = \frac{2\pi}{3}$ // AS REQUIRED

(b) • $\phi = 2\pi - \frac{2\pi}{3} = \frac{4\pi}{3}$

• AREA OF THE MAJOR SECTOR 

$A = \frac{1}{2}r^2\phi^c = \frac{1}{2} \times 12^2 \times \frac{4\pi}{3} = 96\pi$

• AREA OF THE TRIANGLE $A\hat{O}B = \frac{1}{2} \times 12 \times 12 \times \sin(2\theta)$

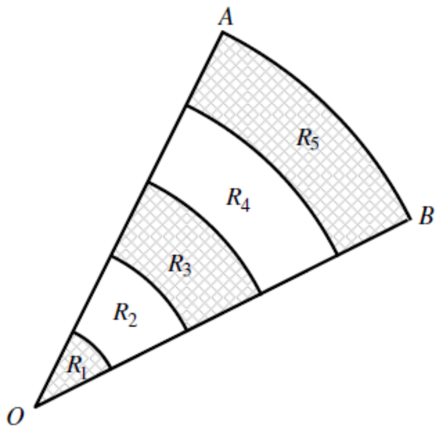
$= \frac{1}{2} \times 144 \times \sin\left(\frac{2\pi}{3}\right)$

$= 36\sqrt{3}$

• AREA OF MAJOR SEGMENT $= 96\pi + 36\sqrt{3}$

$= 12(8\pi + 3\sqrt{3})$ // AS REQUIRED

Question 44 (***+)



The figure above shows a circular sector OAB.

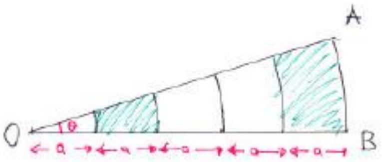
The sides of the sector are equally divided into five equal parts.

Using these divisions arcs are drawn inside the original sector, creating five distinct regions R_1, R_2, R_3, R_4 and R_5, as shown in the figure.

Show that the areas of the regions R_2 and R_5 are in the ratio $1:3$.

proof

AREA OF SECTOR $= \frac{1}{2}r^2\theta$



$$R_2 = \frac{1}{2}(2a)^2\theta - \frac{1}{2}a^2\theta = 2a^2\theta - \frac{1}{2}a^2\theta = \frac{3}{2}a^2\theta$$

$$R_5 = \frac{1}{2}(5a)^2\theta - \frac{1}{2}(4a)^2\theta = \frac{25}{2}a^2\theta - 8a^2\theta = \frac{9}{2}a^2\theta$$

HENCE $$\frac{\text{AREA } R_2}{\text{AREA } R_5} = \frac{\frac{3}{2}a^2\theta}{\frac{9}{2}a^2\theta} = \frac{\frac{3}{2}}{\frac{9}{2}} = \frac{3}{9} = \frac{1}{3}$$

$\therefore$ AREA R_2 : AREA R_5
1 : 3

Parametric

Question 10 (**+)

A curve C is given parametrically by

$$x = 2t + 1, \quad y = \frac{3}{2t}, \quad t \in \mathbb{R}, \quad t \neq 0.$$

a) Find a simplified expression for $\frac{dy}{dx}$ in terms of t.

The point P is the point where C crosses the y axis.

b) Determine the coordinates of P.

c) Find an equation of the tangent to C at P.

$\boxed{\frac{dy}{dx} = -\frac{3}{4t^2}}$, $\boxed{P(0,-3)}$, $\boxed{y = -3x - 3}$

(a) $x = 2t+1$, $y = \frac{3}{2t} = \frac{3}{2}t^{-1}$ $\Big\}$ $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{3}{2}t^{-2}}{2} = -\frac{3}{4} \times \frac{1}{t^2} = -\frac{3}{4t^2}$

(b) when $x=0 \Rightarrow 0 = 2t+1$
$-1 = 2t$
$t = -\frac{1}{2}$

Hence $y = \frac{3}{2(-\frac{1}{2})} = \frac{3}{-1} = -3$

$\therefore (0,-3)$

(c) $\left.\frac{dy}{dx}\right|_{(0,-3)} = \left.\frac{dy}{dx}\right|_{t=-\frac{1}{2}} = -\frac{3}{4(-\frac{1}{2})^2} = -3$

HENCE EQUATION OF THE TANGENT THROUGH $(0,-3)$ $m=-3$

$y - y_0 = m(x - x_0)$
$y+3 = -3(x-0)$
$y = -3x-3$

Question 11 (**+)

A curve known as a cycloid is given by the parametric equations

$$x = 4\theta - \cos\theta\,, \quad y = 1+\sin\theta\,, \quad 0 \le \theta \le 2\pi\,.$$

a) Find an expression for $\frac{dy}{dx}$, in terms of θ.

b) Determine the exact coordinates of the stationary points of the curve.

$\boxed{\frac{dy}{dx} = \frac{\cos\theta}{4+\sin\theta}}$, $\boxed{(2\pi,2),\ (6\pi,0)}$

(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta}{4+\sin\theta}$

(b) FOR STATIONARY POINTS $\frac{dy}{dx} = 0$

$\therefore \frac{\cos\theta}{4+\sin\theta} = 0$

$\cos\theta = 0$

$\theta = \arccos(0) = \frac{\pi}{2}$
$\theta = 2\pi - \arccos 0 = \frac{3\pi}{2}$

• IF $\theta = \frac{\pi}{2}$ $\quad x = 4\left(\frac{\pi}{2}\right) - \cos\frac{\pi}{2} = 2\pi$
$y = 1 + \sin\frac{\pi}{2} = 2$
$\therefore (2\pi, 2)$

• IF $\theta = \frac{3\pi}{2}$ $\quad x = 4\left(\frac{3\pi}{2}\right) - \cos\frac{3\pi}{2} = 6\pi$
$y = 1 + \sin\frac{3\pi}{2} = 0$
$(6\pi, 0)$

Question 12 (***)

A curve C is given parametrically by

$$x = 4t - 1, \quad y = \frac{5}{2t} + 10, \quad t \in \mathbb{R}, \; t \neq 0.$$

The curve C crosses the x axis at the point A.

a) Find the coordinates of A.

b) Show that an equation of the tangent to C at A is

$$10x + y + 20 = 0.$$

c) Determine a Cartesian equation for C.

$$\boxed{(-2,0)}, \quad \boxed{(x+1)(y-10) = 10 \quad \text{or} \quad y = \frac{10(x+2)}{x+1}}$$

(a) $x = 4t - 1$

$y = \frac{5}{2t} + 10 = \frac{5}{2}t^{-1} + 10$

• $y = 0 \Rightarrow 0 = \frac{5}{2t} + 10$

$\Rightarrow -10 = \frac{5}{2t}$

$\Rightarrow -20t = 5$

$\Rightarrow t = -\frac{1}{4}$

$x = 4t - 1$

$x = 4\left(-\frac{1}{4}\right) - 1$

$x = -2$

$\therefore A(-2, 0)$

(b) $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\frac{5}{2}t^{-2}}{4} = -\frac{5}{8} \times \frac{1}{t^2} = -\frac{5}{8t^2}$

$\left.\frac{dy}{dx}\right|_A = \left.\frac{dy}{dx}\right|_{t=-\frac{1}{4}} = -\frac{5}{8\left(-\frac{1}{4}\right)^2} = -\frac{5}{\frac{1}{2}} = -10$

EQUATION OF TANGENT THROUGH $A(-2,0)$ & $m = -10$:

$y - y_0 = m(x - x_0)$

$y - 0 = -10(x+2)$

$y = -10x - 20$

$y + 10x + 20 = 0$

(c) $x = 4t - 1$

$\boxed{x + 1 = 4t}$

$y = \frac{5}{2t} + 10$

$y - 10 = \frac{5}{2t}$

$\boxed{y - 10 = \frac{10}{4t}}$

HENCE

$y - 10 = \frac{10}{x+1}$

$(y-10)(x+1) = 10$

OR $y = \frac{10}{x+1} + 10$

$y = \frac{10 + 10(x+1)}{x+1}$

$y = \frac{10x + 20}{x+1} = \frac{10(x+2)}{x+1}$

Question 22 (***)

A curve is defined by the parametric equations

$$x=\frac{t+3}{t+1}, \quad y=\frac{2}{t+2}, \quad t\in\mathbb{R},\ t\neq -1,\ t\neq -2.$$

Show clearly that …

a) … $\dfrac{dy}{dx}=\left(\dfrac{t+1}{t+2}\right)^2$.

b) … a Cartesian equation for the curve is given by

$$y=\frac{2(x-1)}{x+1}.$$

$\boxed{\text{proof}}$

(a) $x=\dfrac{t+3}{t+1} \Rightarrow \dfrac{dx}{dt}=\dfrac{(t+1)\times 1-(t+3)\times 1}{(t+1)^2}=\dfrac{\cancel{t}+1-\cancel{t}-3}{(t+1)^2}=\dfrac{-2}{(t+1)^2}$

$y=2(t+2)^{-1} \Rightarrow \dfrac{dy}{dt}=-2(t+2)^{-2}=\dfrac{-2}{(t+2)^2}$

$\dfrac{dy}{dx}=\dfrac{dy/dt}{dx/dt}=\dfrac{\frac{-2}{(t+2)^2}}{\frac{-2}{(t+1)^2}}=\dfrac{-2(t+1)^2}{-2(t+2)^2}=\dfrac{(t+1)^2}{(t+2)^2}=\left(\dfrac{t+1}{t+2}\right)^2$ // AS REQUIRED

(b) $y=\dfrac{2}{t+2}$

$\Rightarrow t+2=\dfrac{2}{y}$

$\Rightarrow t+1=\dfrac{2}{y}-1$

$\Rightarrow t+3=\dfrac{2}{y}+1$

HENCE

$\Rightarrow x=\dfrac{\frac{2}{y}+1}{\frac{2}{y}-1}$

$\Rightarrow x=\dfrac{2+y}{2-y}$

$\Rightarrow 2x-xy=2+y$

$\Rightarrow 2x-2=y+xy$

$\Rightarrow 2(x-1)=y(x+1)$

$\Rightarrow \dfrac{2(x-1)}{x+1}=y$ // AS REQUIRED

Question 31 (***+)

A curve is given parametrically by the equations

$$x=3t-2\sin t, \quad y=t^2+t\cos t, \quad 0\leq t<2\pi.$$

Show that an equation of the tangent at the point on the curve where $t=\dfrac{\pi}{2}$ is given by

$$y=\frac{\pi}{6}(x+2).$$

$\boxed{\text{proof}}$

$$x = 3t - 2\sin t \\ y = t^2 + t\cos t \quad \Rightarrow \quad \frac{dx}{dt} = 3 - 2\cos t, \quad \frac{dy}{dt} = 2t + \cos t - t\sin t$$

- $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2t + \cos t - t\sin t}{3 - 2\cos t}$

- $\dfrac{dy}{dx}\Big|_{t=\frac{\pi}{2}} = \dfrac{2\times\frac{\pi}{2} + \cancel{\cos\frac{\pi}{2}} - \frac{\pi}{2}\sin\frac{\pi}{2}}{3 - \cancel{2\cos\frac{\pi}{2}}} = \dfrac{\pi - \frac{\pi}{2}}{3} = \dfrac{\pi}{6}$

- when $t = \frac{\pi}{2}$ $\quad x = 3\left(\frac{\pi}{2}\right) - 2\sin\frac{\pi}{2} = \frac{3\pi}{2} - 2$

  $y = \left(\frac{\pi}{2}\right)^2 + \cancel{\frac{\pi}{2}\cos\frac{\pi}{2}} = \frac{\pi^2}{4}$ $\quad \left(\frac{3\pi}{2} - 2, \frac{\pi^2}{4}\right)$

- EQUATION OF TANGENT: $y - \frac{\pi^2}{4} = \frac{\pi}{6}\left(x - \frac{3\pi}{2} + 2\right)$

$$y - \cancel{\frac{\pi^2}{4}} = \frac{\pi}{6}x - \cancel{\frac{\pi^2}{4}} + \frac{\pi}{3}$$

$$y = \frac{\pi}{6}x + \frac{\pi}{3}$$

$$y = \frac{\pi}{6}(x+2) \quad //$$

Question 34 (***+)

The curve C has parametric equations

$$x = \cos\theta, \ y = \sin 2\theta, \ 0 \le \theta < 2\pi.$$

The point P lies on C where $\theta = \dfrac{\pi}{6}$.

a) Find the gradient at P.

b) Hence show that the equation of the tangent at P is

$$2y + 4x = 3\sqrt{3}.$$

c) Show that a Cartesian equation of C is

$$y^2 = 4x^2\left(1 - x^2\right).$$

$$\boxed{\left.\frac{dy}{dx}\right|_P = -2}$$

(a) $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos 2\theta}{-\sin\theta}$ $\quad \therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{2\cos\frac{\pi}{3}}{-\sin\frac{\pi}{6}} = \frac{2\times\frac{1}{2}}{-\frac{1}{2}} = -2$

(b) when $\theta = \frac{\pi}{6}$, $x = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$
$y = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\therefore$ TANGENT AT $A\left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$, GRADIENT -2

$\Rightarrow y - y_0 = m(x - x_0)$
$\Rightarrow y - \frac{\sqrt{3}}{2} = -2\left(x - \frac{\sqrt{3}}{2}\right)$
$\Rightarrow y - \frac{\sqrt{3}}{2} = -2x + \sqrt{3}$
$\Rightarrow 2y - \sqrt{3} = -4x + 2\sqrt{3}$
$\Rightarrow 4x + 2y = 3\sqrt{3}$

(c) $y = \sin 2\theta$
$\Rightarrow y = 2\sin\theta\cos\theta$
$\Rightarrow y^2 = 4\sin^2\theta\cos^2\theta$
$\Rightarrow y^2 = 4\cos^2\theta(1-\cos^2\theta)$

$\therefore y^2 = 4x^2(1-x^2)$

AS REQUIRED

Question 36 (***+)

A curve C is defined parametrically by

$$x = t + \ln t, \quad y = t - \ln t, \quad t > 0.$$

a) Find the coordinates of the turning point of C.

b) Show that a Cartesian equation for C is

$$4e^{x-y} = (x+y)^2.$$

$\boxed{(1,1)}$

(a) $x = t + \ln t$
$y = t - \ln t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1-\frac{1}{t}}{1+\frac{1}{t}} = \frac{t-1}{t+1}$

SOLVE FOR ZERO $t = 1$

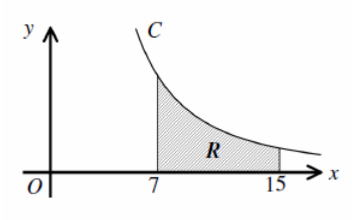
$\therefore$ T.P IS $(1+\ln 1, 1-\ln 1)$ IE $(1,1)$

(b) $x + y = 2t$
$x - y = 2\ln t$

$\Rightarrow \boxed{t = \frac{x+y}{2}}$
$\Rightarrow x - y = 2\ln\left(\frac{x+y}{2}\right)$
$\Rightarrow e^{x-y} = e^{2\ln\left(\frac{x+y}{2}\right)}$
$\Rightarrow e^{x-y} = e^{\ln\left(\frac{x+y}{2}\right)^2}$
$\Rightarrow e^{x-y} = \left(\frac{x+y}{2}\right)^2$
$\Rightarrow e^{x-y} = \frac{(x+y)^2}{4}$
$\Rightarrow 4e^{x-y} = (x+y)^2$

AS REQUIRED

Question 2 (**+)



The figure above shows the curve C, given parametrically by

$$x = 4t - 1, \quad y = \frac{16}{t^2}, \quad t > 0.$$

The finite region R is bounded by C, the x axis and the straight lines with equations $x = 7$ and $x = 15$.

a) Show that the area of R is given by

$$\int_{t_1}^{t_2} \frac{64}{t^2}\, dt,$$

stating the values of t_1 and t_2.

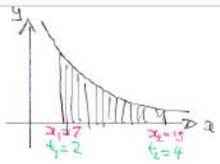
b) Hence find the area of R.

c) Find a Cartesian equation of C, in the form $y = f(x)$.

d) Use the Cartesian equation of C to verify the result of part (b).

$\boxed{t_1 = 2}$, $\boxed{t_2 = 4}$, $\boxed{\text{area} = 16}$, $\boxed{y = \dfrac{256}{(x+1)^2}}$

(a)



• $x = 4t - 1$
$7 = 4t - 1$
$8 = 4t$
$\boxed{t = 2}$

• $x = 4t - 1$
$15 = 4t - 1$
$16 = 4t$
$\boxed{t = 4}$

$$A = \int_{x_1}^{x_2} y(x)\, dx = \int_{t_1}^{t_2} y(t) \frac{dx}{dt}\, dt = \int_{2 \leftarrow t_1}^{4 \leftarrow t_2} \left(\frac{16}{t^2}\right)(4)\, dt$$

$$\therefore A = \int_2^4 \frac{64}{t^2}\, dt$$ // AS REQUIRED

(b) $A = \left[-\frac{64}{t}\right]_2^4 = \left[\frac{64}{t}\right]_4^2 = 32 - 16 = 16$ //

(c) $\left.\begin{array}{l} x = 4t - 1 \\ x + 1 = 4t \\ (x+1)^2 = 16t^2 \end{array}\right\} \Rightarrow y = \frac{16}{t^2}$

$y = \frac{256}{16t^2}$

$y = \frac{256}{(x+1)^2}$ //

(d) $A = \int_7^{15} 256(x+1)^{-2}\, dx$

$A = \left[-256(x+1)^{-1}\right]_7^{15}$

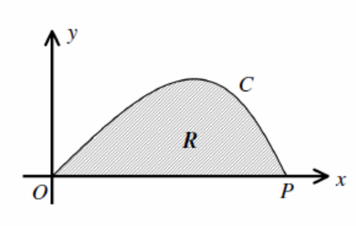
$A = \left[\frac{256}{x+1}\right]_{15}^{7}$

$A = \frac{256}{8} - \frac{256}{16}$

$A = 32 - 16 = 16$ //

AS BEFORE.

Question 7 (***)



The figure above shows the curve C, given parametrically by

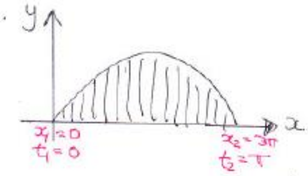
$$x = 3t + \sin t, \quad y = 2\sin t \ , \quad 0 \le t \le \pi \ .$$

The curve meets the coordinate axes at the point P and at the origin O.

The finite region R is bounded by C and the x axis.

Determine the area of R.

$\boxed{}$, $\boxed{\text{area} = 12}$



VERIFY t

- $t=0 \quad x=0, y=0 \quad \therefore (0,0)$
- $t=\pi \quad x=3\pi, y=0 \quad \therefore P(\frac{3\pi}{2},0)$

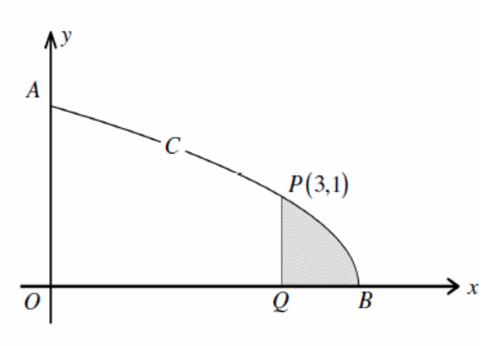
$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt = \int_0^{\pi} (2\sin t)\underbrace{(3+\cos t)}_{\frac{dx}{dt}}$$

$$= \int_0^{\pi} 6\sin t + 2\sin t\cos t \,dt = \int_0^{\pi} 6\sin t + \sin 2t \,dt$$

$$= \left[-6\cos t - \tfrac{1}{2}\cos 2t\right]_0^{\pi} = \left[6\cos t + \tfrac{1}{2}\cos 2t\right]_{\pi}^{0}$$

$$= \left(6+\tfrac{1}{2}\right) - \left(-6+\tfrac{1}{2}\right) = 6 + \cancel{\tfrac{1}{2}} + 6 - \cancel{\tfrac{1}{2}} = 12$$

Question 9 (***+)



The figure above shows the curve C, with parametric equations

$$x = 4\sin^2 t\ ,\ \ y = 2\cos t\ ,\ 0 \le t \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B.

The point $P(3,1)$ lies on C.

The point Q lies on the x axis so that PQ is parallel to the y axis.

a) Show that the area of the shaded region bounded by C, the line PQ and the x axis is given by the integral

$$16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 t \sin t \ \ dt .$$

b) Evaluate the above integral to find the area of the shaded region.

$\boxed{}$, $\boxed{\text{area} = \frac{2}{3}}$

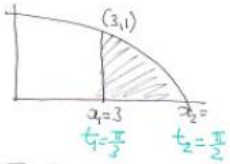
(a) $x = 4\sin^2 t$ $\quad 0 \le t \le \frac{\pi}{2}$
$y = 2\cos t$

• when $y = 0$ $\quad 2\cos t = 0$
$\cos t = 0$
$\boxed{t = \frac{\pi}{2}}$

• $3 = 4\sin^2 t$
$\frac{3}{4} = \sin^2 t$
$\sin t = \sqrt{\frac{3}{4}}$ or $-\sqrt{\frac{3}{4}}$
$\boxed{t = \frac{\pi}{3} \text{ or } \cancel{-\frac{\pi}{3}}}$

• HENCE
$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{t_1}^{t_2} y(t)\frac{dx}{dt}\,dt$

$\text{AREA} = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (2\cos t)(8\sin t\cos t)\,dt$



$\therefore$ $\text{AREA} = 16\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2 t\sin t\,dt$ AS REQUIRED

(b) $= \left[-\frac{16}{3}\cos^3 t\right]_{\pi/3}^{\pi/2}$ (BY INSPECTION)

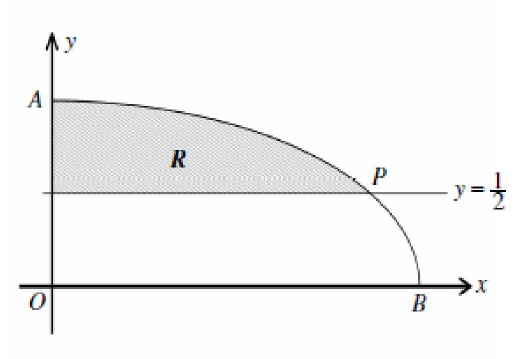
$= \frac{16}{3}\left[\cos^3 t\right]_{\pi/2}^{\pi/3}$

$= \frac{16}{3}\left[\cos^3\frac{\pi}{3} - \cancel{\cos^3\frac{\pi}{2}}\right]$

$= \frac{16}{3} \times \frac{1}{8}$

$= \frac{2}{3}$

Question 16 (****)



The figure above shows the curve C, with parametric equations

$$x = 4\cos\theta, \quad y = \sin\theta \, , \; 0 \le \theta \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B. The straight line with equation $y = \frac{1}{2}$ meets C at the point P.

a) Show that the area under the arc of the curve between A and P, and the x axis, is given by the integral

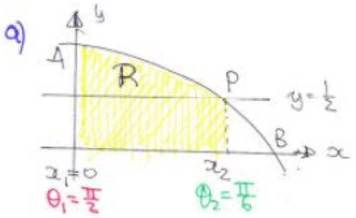
$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \; d\theta .$$

The shaded region R is bounded by C, the straight line with equation $y = \frac{1}{2}$ and the y axis.

b) Find an exact value for the area of R.

$$\boxed{\text{area} = \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)}$$

a)



- $x = 4\cos\theta$
  $0 = 4\cos\theta$
  $\cos\theta = 0$
  $\theta = \pi/2$
  (ONLY SOLUTION IN RANGE)

- $y = \sin\theta$
  $\frac{1}{2} = \sin\theta$
  $\theta = \frac{\pi}{6}$
  ONLY SOLUTION IN RANGE

$$\text{AREA} = \int_{x_1}^{x_2} y(x)\,dx = \int_{\theta_1}^{\theta_2} y(\theta)\frac{dx}{d\theta}\,d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} (\sin\theta)(-4\sin\theta)\,d\theta = \int_{\frac{\pi}{2}}^{\frac{\pi}{6}} -4\sin^2\theta\,d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta\,d\theta$$

AS REQUIRED

b) FINISH THE INTEGRATION ...

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2 - 2\cos 2\theta = \left[2\theta - \sin 2\theta\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= (\pi - 0) - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}$$

THE VALUE OF θ AT P IS $\frac{\pi}{6}$ (FOUND EARLIER)
$x = 4\cos\theta$
$x = 4\cos\frac{\pi}{6}$
$x = 2\sqrt{3}$

$$\Longrightarrow \boxed{\sqrt{3}}\ \frac{1}{2}$$

$2\sqrt{3}$

$\therefore$ REQUIRED AREA

$$= \left(\frac{2\pi}{3} + \frac{\sqrt{3}}{2}\right) - \sqrt{3}$$

$$= \frac{2\pi}{3} - \frac{1}{2}\sqrt{3}$$

$$= \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)$$

Numerical methods

Question 5 (**+)

$$x^3 - x^2 = 6x + 6, \quad x \in \mathbb{R}.$$

a) Show that the above equation has a root α in the interval $(3,4)$.

b) Show that the above equation can be written as

$$x = \sqrt{\frac{6x+6}{x-1}}.$$

An iterative formula of the form given in part (**b**), starting with x_0 is used to find α.

c) Give two different values for x_0 that would not produce an answer for x_1.

d) Starting with $x_0 = 3.3$ find the value of x_1, x_2, x_3 and x_4, giving each of the answers correct to 3 decimal places.

e) By considering the sign of an appropriate function in a suitable interval, show clearly that $\alpha = 3.33691$, correct to 5 decimal places.

$\boxed{}$, $\boxed{x_0 \neq 1,\ 0,\ 0.5 \text{ etc}}$, $\boxed{x_1 = 3.349,\ x_2 = 3.333,\ x_3 = 3.338,\ x_4 = 3.336}$

(a) $x^3 - x^2 = 6x + 6$

$x^3 - x^2 - 6x - 6 = 0$

$f(x) = x^3 - x^2 - 6x - 6$

$f(3) = -6$

$f(4) = 18$

AS $f(x)$ IS CONTINUOUS AND CHANGES SIGN THERE MUST BE A SOLUTION IN THE INTERVAL $(3,4)$

(b) $x^3 - x^2 = 6x + 6$

$x^2(x-1) = 6x + 6$

$x^2 = \frac{6x+6}{x-1}$

$x = \pm\sqrt{\frac{6x+6}{x-1}}$

$x = \sqrt{\frac{6x+6}{x-1}}$

(c) $x_0 = 1$ (UNDEFINED)

$x_0 = 0$ (NEGATIVE IN THE SQUARE ROOT)

$x_0 = 3.3$

$x_1 = 3.349$

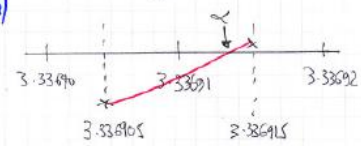
$x_2 = 3.333$

$x_3 = 3.338$

$x_4 = 3.336$

$x_{n+1} = \sqrt{\frac{6x_n+6}{x_n-1}}$

(d)



$f(3.336905) = -0.00014$

$f(3.336915) = 0.000063$

CHANGE OF SIGN $\Rightarrow 3.336905 < \alpha < 3.336915$

$\Rightarrow$ ROOT $\alpha = 3.33691$

5 d.p

Question 7 (***)

$$x^3 + 3x = 5, \ x \in \mathbb{R}.$$

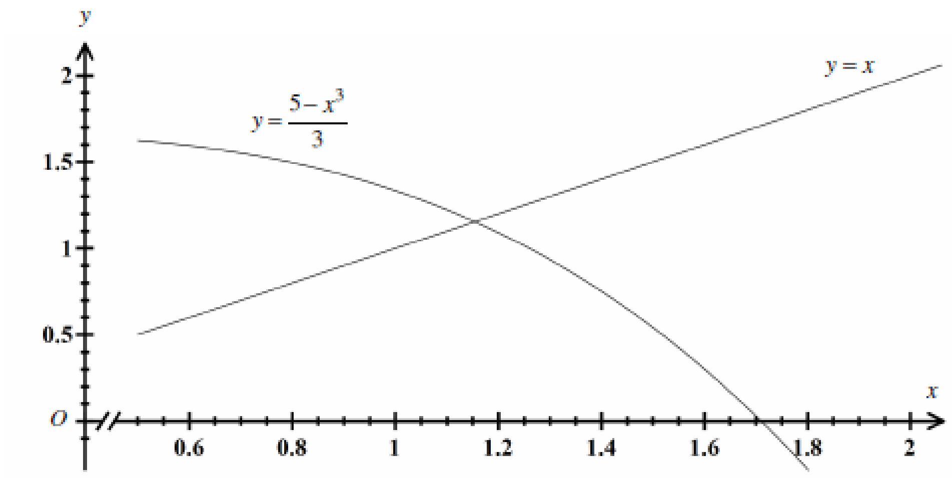
a) Show that the above equation has a root α between 1 and 2.

An attempt is made to find α using the iterative formula

$$x_{n+1} = \frac{5 - x_n^{\ 3}}{3}, \ x_1 = 1.$$

b) Find, to 2 decimal places, the value of x_2, x_3, x_4, x_5 and x_6.

The diagram below is used to investigate the results of these iterations.



c) On a copy of this diagram draw a "staircase" or "cobweb" pattern marking the position of x_1, x_2, x_3 and x_4, further stating the results of these iterations.

d) Use the iterative formula

$$x_{n+1} = \sqrt[3]{5-3x_n}\ ,\ x_1 = 1,$$

to find, to 2 decimal places, the value of x_2, x_3, x_4, x_5 and x_6.

$\boxed{}$, $\boxed{x_2 = 1.33,\quad x_3 = 0.88,\quad x_4 = 1.44,\quad x_5 = 0.67,\quad x_6 = 1.57}$,

$\boxed{x_2 = 1.26,\quad x_3 = 1.07,\quad x_4 = 1.22,\quad x_5 = 1.11,\quad x_6 = 1.19}$

(a) $x^3 + 3x = 5$

$x^3 + 3x - 5 = 0$

LET $f(x) = x^3 + 3x - 5$

$f(1) = 1 + 3 - 5 = -1$

$f(2) = 8 + 6 - 5 = 9$

AS $f(x)$ IS CONTINUOUS & CHANGES SIGN, THERE MUST BE A ROOT BETWEEN 1 & 2

(b) $x_{n+1} = \dfrac{5 - x_n^3}{3}$

$x_1 = 1$

$x_2 = 1.33$

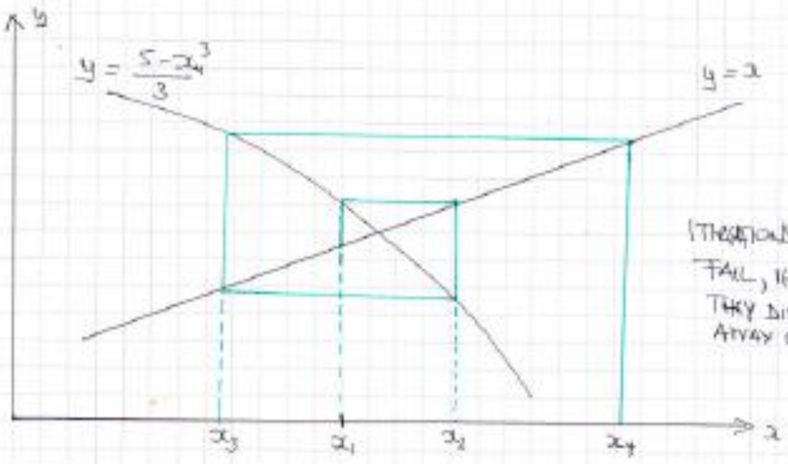
$x_3 = 0.88$

$x_4 = 1.44$

$x_5 = 0.67$

$x_6 = 1.57$

(c)



ITERATIONS FAIL, IE THEY DIVERGE AWAY FROM TH

(d) $x_{n+1} = \sqrt[3]{5 - 3x_n}$

$x_1 = 1$

$x_2 = 1.26$

$x_3 = 1.07$

$x_4 = 1.22$

$x_5 = 1.11$

$x_6 = 1.19$

Question 8 (***)

$$x^3 = 5x + 1, \ x \in \mathbb{R}.$$

a) Show that the above equation has a root α between 2 and 3.

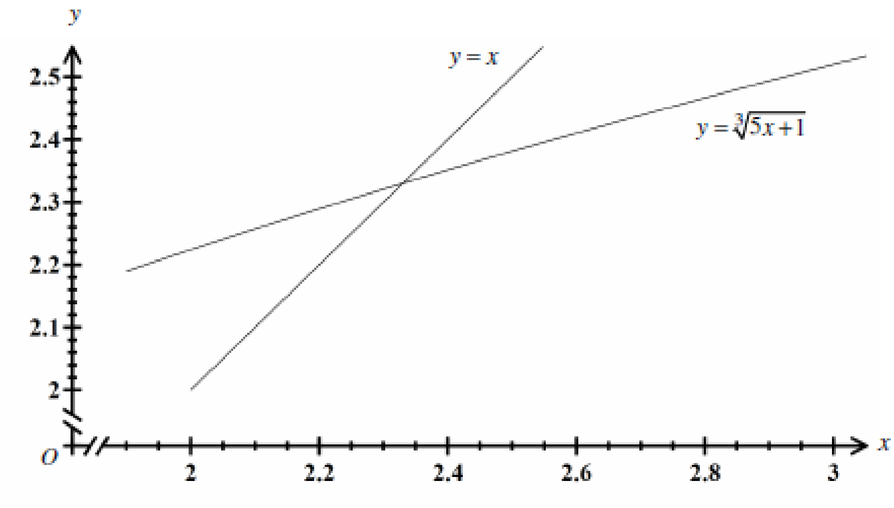
The iterative formula

$$x_{n+1} = \sqrt[3]{5x_n + 1}, \ x_1 = 2,$$

is to be used to find α

b) Find, to 2 decimal places, the value of x_2, x_3 and x_4.

The diagram below is used to investigate the results of these iterations.



c) On a copy of this diagram draw a "staircase" or "cobweb" pattern showing how these iterations converge to α, marking the position of x_1, x_2, x_3 and x_4.

$\boxed{}$, $\boxed{x_2 = 2.22, \quad x_3 = 2.30, \quad x_4 = 2.32}$

(a) $x^3 = 5x + 1$

$x^3 - 5x - 1 = 0$

$f(x) = x^3 - 5x - 1$

$f(2) = 8 - 10 - 1 = -3$

$f(3) = 27 - 15 - 1 = 11$

AS $f(x)$ IS CONTINUOUS & CHANGES SIGN THERE MUST BE A SOLUTION BETWEEN 2 & 3

(b) $x_{n+1} = \sqrt[3]{5x_n + 1}$

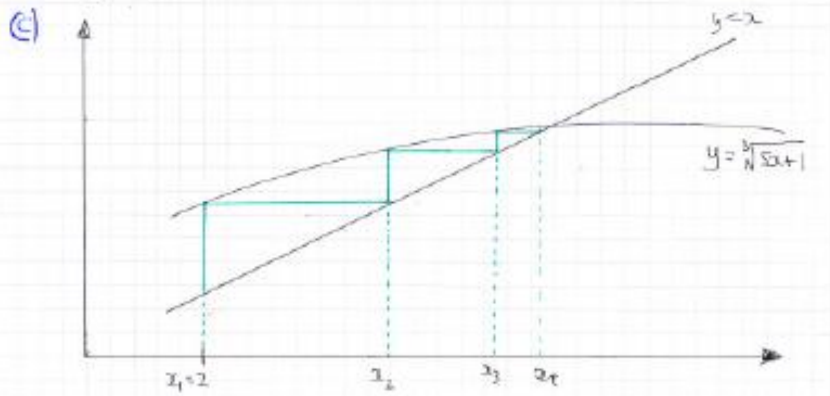
$x_1 = 2$

$x_2 = 2.22$

$x_3 = 2.30$

$x_4 = 2.32$

(c)



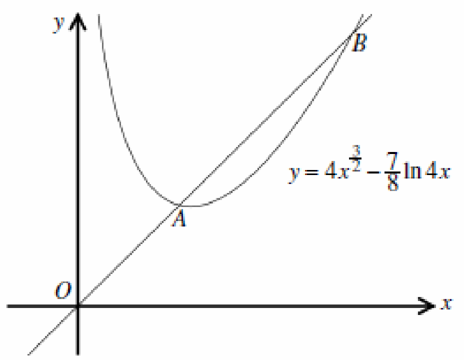
Question 17 (***+)

A curve C has equation

$$y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x, \quad x \in \mathbb{R}, \quad x > 0.$$

The point A is on C, where $x = \frac{1}{4}$.

a) Find an equation of the normal to the curve at A.



This normal meets the curve again at the point B, as shown in the figure above.

b) Show that the x coordinate of B satisfies the equation

$$x = \left(\frac{16x + 7\ln 4x}{32}\right)^{\frac{2}{3}}.$$

The recurrence relation

$$x_{n+1} = \left(\frac{16x_n + 7\ln 4x_n}{32}\right)^{\frac{2}{3}}, \quad x_0 = 0.7$$

is to be used to find the x coordinate of B.

c) Find, to 3 decimal places, the value of x_1, x_2, x_3 and x_4.

d) Show that the x coordinate of B is 0.6755, correct to 4 decimal places.

$\boxed{}$, $\boxed{y = 2x}$, $\boxed{x_1 = 0.692, \quad x_2 = 0.686, \quad x_3 = 0.683, \quad x_4 = 0.680}$

a) $y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$\frac{dy}{dx} = 6x^{\frac{1}{2}} - \frac{7}{8} \times \frac{1}{x}$

$\left.\frac{dy}{dx}\right|_{x=\frac{1}{4}} = 6 \times \frac{1}{2} - \frac{7}{8} \times 4 = 3 - \frac{7}{2} = -\frac{1}{2}$

- NORMAL GRADIENT $= 2$
- $y = 4 \times \left(\frac{1}{4}\right)^{\frac{3}{2}} - \cancel{\frac{7}{8}\ln 1} = \frac{1}{2}$ IF $\left(\frac{1}{4}, \frac{1}{2}\right)$

$y - y_0 = m(x - x_0)$

$y - \frac{1}{2} = 2\left(x - \frac{1}{4}\right)$

$y \cancel{- \frac{1}{2}} = 2x \cancel{- \frac{1}{2}}$

$y = 2x$

b) $y = 2x$ & $y = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x + \frac{7}{8}\ln 4x = 4x^{\frac{3}{2}}$

$\frac{16x + 7\ln 4x}{8} = 4x^{\frac{3}{2}}$

$x^{\frac{3}{2}} = \frac{16x + 7\ln 4x}{32}$

$x = \left(\frac{16x + 7\ln 4x}{32}\right)^{\frac{2}{3}}$

c) $x_{n+1} = \left[\frac{16x_n + 7\ln(4x_n)}{32}\right]^{\frac{2}{3}}$

$x_0 = 0.7$

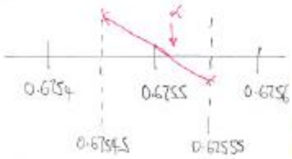
$x_1 \simeq 0.692$

$x_2 \simeq 0.686$

$x_3 \simeq 0.683$

$x_4 \simeq 0.680$

d)



EQUATION SOLVED

$2x = 4x^{\frac{3}{2}} - \frac{7}{8}\ln 4x$

$2x - 4x^{\frac{3}{2}} + \frac{7}{8}\ln 4x = 0$

(LET $f(x) = 2x - 4x^{\frac{3}{2}} + \frac{7}{8}\ln 4x$

$f(0.67545) = 0.000083 > 0$

$f(0.67555) = -0.000080 < 0$

$\Rightarrow$ CHANGE OF SIGN (AND CONTINUITY) IMPLY

$0.67545 < \alpha < 0.6755$

$\alpha \simeq 0.6755$

CORRECT TO 4 d.p.

Question 19 (****)

The curve C has equation

$$y = \frac{3x+1}{x^3 - x^2 + 5}.$$

The curve has a single turning point at M, with approximate coordinates $(1.4, 0.9)$.

a) Show that the x coordinate of M is a solution of the equation

$$x = \sqrt[3]{\frac{1}{3}x + \frac{5}{2}}.$$

b) By using an iterative formula based on the equation of part (a), determine the coordinates of M correct to three decimal places.

$\boxed{}$, $\boxed{M(1.439, 0.900)}$

(a) $y = \dfrac{3x+1}{x^3-x^2+5}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(x^3-x^2+5)\times 3 - (3x+1)(3x^2-2x)}{(x^3-x^2+5)^2}$

FOR T.P.s, $\dfrac{dy}{dx} = 0$, IF NUMERATOR ZERO

$\Rightarrow 3x^3 - 3x^2 + 15 - 9x^3 + 6x^2 - 3x^2 + 2x = 0$

$\Rightarrow -6x^3 + 2x + 15 = 0$

$\Rightarrow 6x^3 - 2x - 15 = 0$

$\Rightarrow 6x^3 = 2x + 15$

$\Rightarrow x^3 = \frac{1}{3}x + \frac{5}{2}$

$\Rightarrow x = \sqrt[3]{\frac{1}{3}x + \frac{5}{2}}$

AS REQUIRED

(b) $x_{n+1} = \sqrt[3]{\frac{1}{3}x_n + \frac{5}{2}}$

$x_1 = 1.4...$

$x_2 = 1.43689...$

$x_3 = 1.43887...$

$x_4 = 1.43898...$

$x_5 = 1.43898...$

$y = \dfrac{3(1.43...)+1}{(1.43...)^3 - (1.43...)^2 + 5}$

$y = 0.89981....$

$\therefore M(1.439, 0.900)$

Question 1 (***+)

It is know that the cubic equation below has a root α, which is close to 1.25.

$$x^3 + x = 3.$$

Use an iterative formula based on the Newton Raphson method to find the value of α, correct to 6 decimal places.

$\boxed{\alpha \approx 1.213411}$

$$x^3 + x = 3$$

$$x^3 + x - 3 = 0$$

LET $f(x) = x^3 + x - 3$

$$f'(x) = 3x^2 + 1$$

BY NEWTON-RAPHSON

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 3}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{3x_n^3 + \cancel{x_n} - x_n^3 - \cancel{x_n} + 3}{3x_n^2 + 1}$$

$$x_{n+1} = \frac{2x_n^3 + 3}{3x_n^2 + 1}$$

STARTING WITH $x_0 = 1.25$

$x_1 = 1.214285714$

$x_2 = 1.213412176$

$x_3 = 1.213411663$

$x_4 = 1.213411663$

$\therefore \alpha = 1.213411$

Question 2 (****)

A curve C has equation

$$y = e^{-x} \ln x, \quad x > 0.$$

a) Show that the x coordinate of the stationary point of C is between 1 and 2.

b) Use an iterative formula based on the Newton Raphson method to find the coordinates of the stationary point of C, correct to 8 decimal places.

$\boxed{(1.76322283, 0.09726013)}$

a) $y = e^{-x} \ln x, \; x > 0$

$\frac{dy}{dx} = -e^{-x} \ln x + e^{-x} \times \frac{1}{x} = e^{-x}\left(-\ln x + \frac{1}{x}\right)$

- SOLVE FOR ZERO

$-\ln x + \frac{1}{x} = 0 \quad (e^{-x} \neq 0)$

LET

$f(x) = \frac{1}{x} - \ln x$

$\left.\begin{array}{l} f(1) = 1 > 0 \\ f(2) = -0.193 < 0 \end{array}\right\}$ AS $f(x)$ IS CONTINUOUS BETWEEN 1 & 2 THERE EXISTS AT LEAST ONE VALUE IN THE INTERVAL FOR WHICH $f(x) = 0$, SINCE THE FUNCTION CHANGES SIGN

b) $f(x) = \frac{1}{x} - \ln x$

$f'(x) = -\frac{1}{x^2} - \frac{1}{x}$

BY NEWTON-RAPHSON $\quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$\Rightarrow x_{n+1} = x_n - \frac{\frac{1}{x_n} - \ln x_n}{-\frac{1}{x_n^2} - \frac{1}{x_n}}$

$\Rightarrow x_{n+1} = x_n + \frac{\frac{1}{x_n} - \ln x_n}{\frac{1}{x_n^2} + \frac{1}{x_n}}$

$\Rightarrow x_{n+1} = x_n + \frac{x_n - x_n^2 \ln x_n}{1 + x_n}$

$\Rightarrow x_{n+1} = \frac{x_n + x_n^2 + x_n - x_n^2 \ln x_n}{1 + x_n}$

$\Rightarrow x_{n+1} = \frac{2x_n + x_n^2(1 - \ln x_n)}{1 + x_n}$

Now $\quad x_1 = 1.5$

$x_2 = 1.735081403\ldots$

$x_3 = 1.76291591\ldots$

$x_4 = 1.763222798\ldots$

$x_5 = 1.763222834$

$x_6 = 1.763222834$

$\therefore x \simeq 1.76322283$

$y \simeq e^{-1.763\ldots} \ln(1.763\ldots) \simeq 0.0972601373\ldots$

$\therefore P(1.76322283, 0.09726013)$ 8 d.p.

Functions

Question 8 (**+)

The functions f and g satisfy

$$f(x) = 1 + \tfrac{1}{2}\ln(x+3), \; x \in \mathbb{R}, \; x > 3.$$

$$g(x) = e^{2(x-1)} - 3, \; x \in \mathbb{R}.$$

a) Find, in its **simplest** form, an expression for $fg(x)$.

b) Hence, or otherwise, write down an expression for $f^{-1}(x)$.

$\boxed{fg(x) = x}$, $\boxed{f^{-1}(x) = e^{2(x-1)} - 3}$

(a) $f(g(x)) = f\left(e^{2(x-1)} - 3\right) = 1 + \frac{1}{2}\ln\left[\left(e^{2(x-1)} - 3\right) + 3\right]$

$= 1 + \frac{1}{2}\ln\left(e^{2(x-1)}\right) = 1 + \frac{1}{\cancel{2}} \times \cancel{2}(x-1)$

$= \cancel{1} + x - \cancel{1} = x$

(b) SINCE $f(g(x)) = x \quad g(x) = f^{-1}(x)$

$\therefore f^{-1}(x) = e^{2(x-1)} - 3$

Question 14 (***)

The functions f and g are given by

$$f(x)=\frac{2x+3}{2x-3},\ x\in\mathbb{R},\ x\neq\frac{3}{2}.$$

$$g(x)=x^2+2,\ x\in\mathbb{R}.$$

a) State the range of $g(x)$.

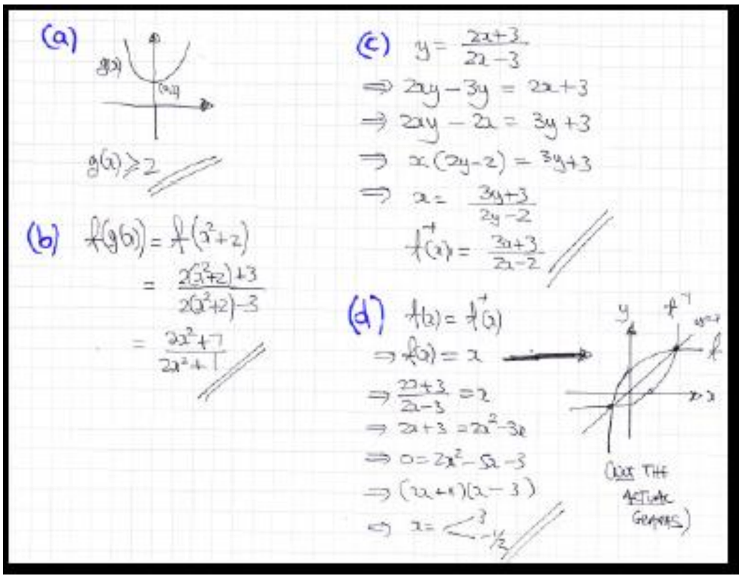
b) Find an expression, as a simplified algebraic fraction, for $fg(x)$.

c) Determine an expression, as a simplified algebraic fraction, for $f^{-1}(x)$.

d) Solve the equation

$$f^{-1}(x)=f(x).$$

$\boxed{}$, $\boxed{g(x)\geq 2}$, $\boxed{fg(x)=\frac{2x^2+7}{2x^2+1}}$, $\boxed{f^{-1}(x)=\frac{3x+3}{2x-2}}$, $\boxed{x=-\frac{1}{2},3}$



Question 15 (***)

$$f(x) = e^{2x} - 4, \ x \in \mathbb{R}.$$

$$g(x) = \frac{1}{x-11}, \ x \in \mathbb{R}, \ x \neq 11.$$

a) Determine the range of $f(x)$.

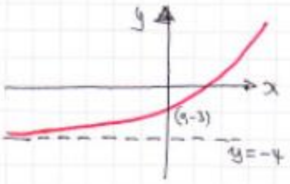
b) Find an expression for the inverse function $f^{-1}(x)$.

c) Solve the equation

$$gf(x) = 1.$$

$\boxed{f(x) > -4}$, $\boxed{f^{-1}(x) = \tfrac{1}{2}\ln(x+4)}$, $\boxed{x = 2\ln 2}$

(a)



$f(x) > -4$

(b) $y = e^{2x} - 4$

$\Rightarrow y + 4 = e^{2x}$

$\Rightarrow 2x = \ln(y+4)$

$\Rightarrow x = \frac{1}{2}\ln(y+4)$

$\therefore f^{-1}(x) = \frac{1}{2}\ln(x+4)$

(c) $g(f(x)) = 1$

$\Rightarrow g\left(e^{2x} - 4\right) = 1$

$\Rightarrow \dfrac{1}{e^{2x} - 4 - 11} = 1$

$\Rightarrow \dfrac{1}{e^{2x} - 15} = 1$

$\Rightarrow 1 = e^{2x} - 15$

$\Rightarrow 16 = e^{2x}$

$\Rightarrow 2x = \ln 16$

$\Rightarrow 2x = 4\ln 2$

$\Rightarrow x = 2\ln 2$

Question 33 (***)

The function f is defined as

$$f : x \mapsto \frac{1}{x+2} + \frac{2x+11}{2x^2+x-6}\quad x \in \mathbb{R},\ x > \frac{3}{2}.$$

a) Show clear that

$$f : x \mapsto \frac{4}{2x-3},\quad x \in \mathbb{R},\ x > \frac{3}{2}.$$

b) Find an expression for f^{-1}, in its simplest form.

c) Find the domain of f^{-1}.

The function g is given by

$$g : x \mapsto \ln(x-1),\ x \in \mathbb{R},\ x > 1.$$

d) Show that $x = 1 + \sqrt{e}$ is the solution of the equation

$$fg(x) = -2.$$

$\boxed{}$, $\boxed{f^{-1}(x) = \frac{3x+4}{2x}}$, $\boxed{x > 0}$

(a) $f(x) = \frac{1}{x+2} + \frac{2x+11}{2x^2+x-6} = \frac{1}{x+2} + \frac{2x+11}{(x+2)(2x-3)} = \frac{2x-3+2x+11}{(x+2)(2x-3)}$

$= \frac{4x+8}{(x+2)(2x-3)} = \frac{4(x+2)}{(x+2)(2x-3)} = \frac{4}{2x-3}$ // AS REQUIRED

(b) $y = \frac{4}{2x-3}$

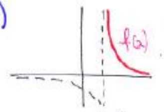
$\Rightarrow 2xy - 3y = 4$

$\Rightarrow 2xy = 3y + 4$

$\Rightarrow x = \frac{3y+4}{2y}$

$\therefore f^{-1}(x) = \frac{3x+4}{2x}$ //

(c)



Range of $f(x)$ is $f(x) > 0$

HENCE DOMAIN OF $f^{-1}(x)$ IS $x > 0$ //

(d) $f(g(x)) = -2$

$f(\ln(x-1)) = -2$

$\frac{4}{2\ln(x-1)-3} = -2$

$4 = 6 - 4\ln(x-1)$

$4\ln(x-1) = 2$

$\ln(x-1) = \frac{1}{2}$

$x - 1 = e^{\frac{1}{2}}$

$x = e^{\frac{1}{2}} + 1$ //

AS REQUIRED

Question 34 (***)

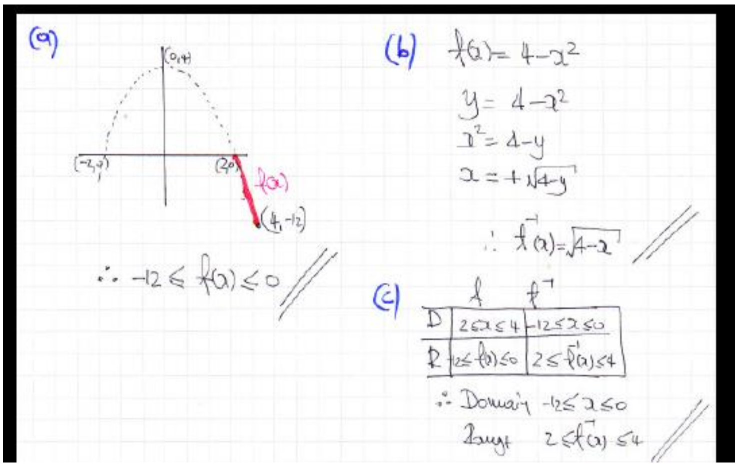
$$f(x) = 4 - x^2,\ x \in \mathbb{R},\ 2 \le x \le 4.$$

a) Determine the range of $f(x)$.

b) Find an expression for the inverse function $f^{-1}(x)$.

c) State the domain and range of $f^{-1}(x)$.

$\boxed{-12 \le f(x) \le 0}$, $\boxed{f^{-1}(x) = \sqrt{4-x}}$, $\boxed{-12 \le x \le 0}$, $\boxed{2 \le f^{-1}(x) \le 4}$



Question 39 (***+)

The function $f(x)$ is defined by

$$f(x)=\sqrt{x+1},\ x\in\mathbb{R},\ x\geq 0.$$

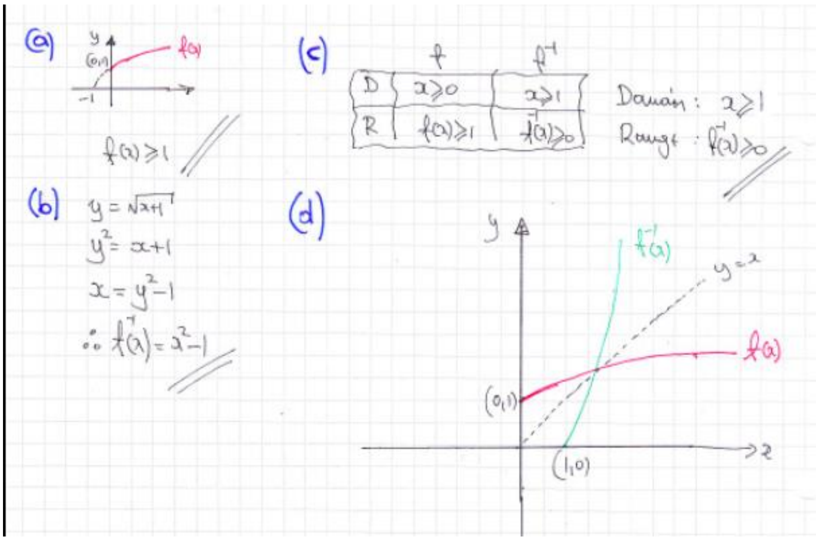
a) Find the range of $f(x)$.

b) Find an expression for $f^{-1}(x)$ in its simplest form.

c) State the domain and range of $f^{-1}(x)$.

d) Sketch in the same diagram $f(x)$ and $f^{-1}(x)$.

$\boxed{f(x)\geq 1}$, $\boxed{f^{-1}(x)=x^2-1}$, $\boxed{x\geq 1,\ \ f^{-1}(x)\geq 0}$



Question 54 (***+)

$$f(x) = e^x, \ x \in \mathbb{R}, \ x > 0.$$

$$g(x) = 2x^3 + 11, \ x \in \mathbb{R}.$$

a) Find and simplify an expression for the composite function $gf(x)$.

b) State the domain and range of $gf(x)$.

c) Solve the equation

$$gf(x) = 27.$$

The equation $gf(x) = k$, where k is a constant, has solutions.

d) State the range of the possible values of k.

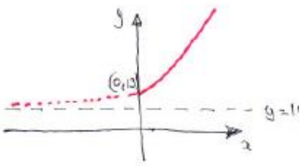
$\boxed{}$, $\boxed{gf(x) = 2e^{3x} + 11}$, $\boxed{x > 0, \ gf(x) > 13}$, $\boxed{x = \ln 2}$, $\boxed{k > 13}$

(a) $g(f(x)) = g(e^x) = 2(e^x)^3 + 11 = 2e^{3x} + 11$

(b) $\boxed{f}$ $\qquad \boxed{g}$

IN $\quad$ OUT $\quad$ IN $\quad$ OUT

$x > 0 \qquad x \in \mathbb{R}$



∴ DOMAIN : $x > 0$

RANGE : $g(f(x)) > 13$

(c) $g(f(x)) = 27$

$2e^{3x} + 11 = 27$

$2e^{3x} = 16$

$e^{3x} = 8$

$3x = \ln 8$

$3x = 3\ln 2$

$x = \ln 2$

(d) FROM GRAPH ABOVE

$g(f(x)) = k$

WILL HAVE SOLUTIONS, IF $k > 13$

Question 71 (****)

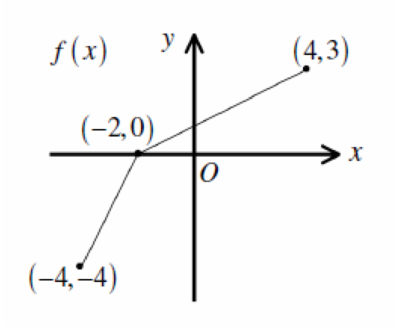


figure 1

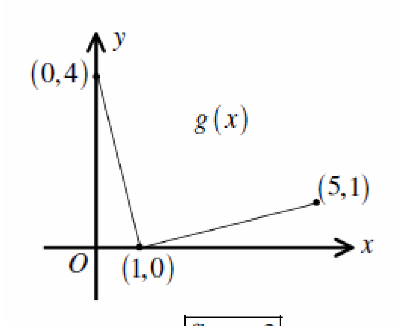


figure 2

Figure 1 and figure 2 above, show the graphs of two piecewise continuous functions $f(x)$ and $g(x)$, respectively.

Each graph consists of two straight line segments joining the points with the coordinates shown in each figure.

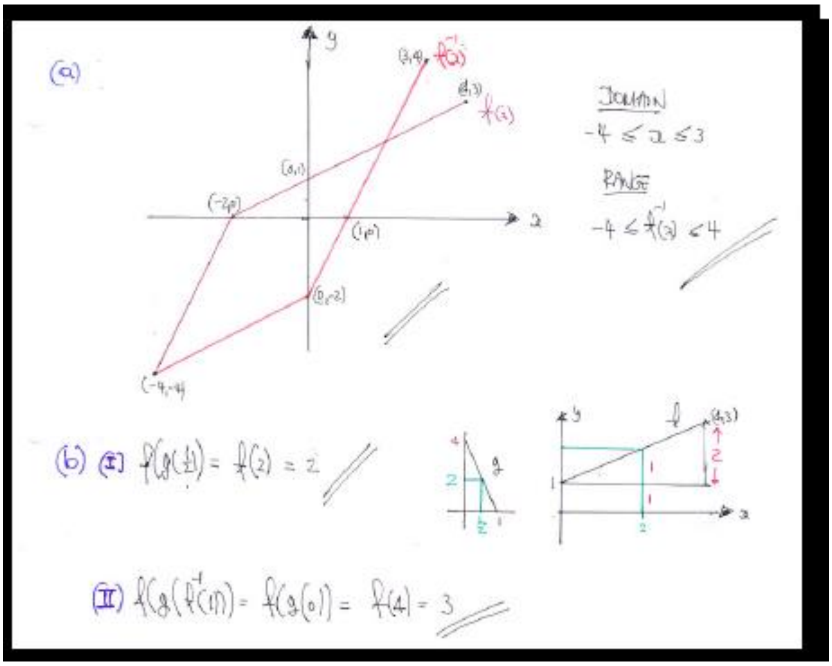
a) Sketch on the same set of axes the graphs of $f(x)$ and its inverse $f^{-1}(x)$, stating the domain and range of $f^{-1}(x)$.

b) Evaluate …

i. … $fg\left(\frac{1}{2}\right)$.

ii. … $fgf^{-1}(1)$.

$\boxed{}$, $\boxed{-4 \le x \le 3}$, $\boxed{-4 \le f^{-1}(x) \le 4}$, $\boxed{fg\left(\frac{1}{2}\right) = 2}$, $\boxed{fgf^{-1}(1) = 3}$



Proof

Exercise 7A

1 Prove that the sum of the first n terms of the arithmetic series with first term a and common difference d is S, where

$$S = \frac{n}{2}[2a + d(n-1)]$$

2 In $\triangle ABC$, $AB = 7\,\text{cm}$, $AC = 5\,\text{cm}$ and $BC = x\,\text{cm}$. Let $\angle BAC$ be denoted by θ. Given that the area of $\triangle ABC$ is $10\,\text{cm}^2$, prove that $\sin\theta = \frac{4}{7}$.
Hence find the two possible sizes of θ in degrees to one decimal place.

3 Prove that for all real values of x

$$x^2 + 1 \geqslant 2x$$

5 Explain why each of the following assertions is not necessarily correct:

(a) $\{\sin x = \frac{3}{5}\} \Rightarrow \{\cos x = \frac{4}{5}\}$

(b) $\{x^4 = 16\} \Rightarrow \{x = -2\}$

(c) $\{x^2 - 5x = 14\} \Rightarrow \{x = -2\}$

8 The $\triangle ABC$ is such that A is (1, 1), B is (−2, 5) and C is (4, 5). Prove that $\triangle ABC$ is isosceles and determine its area.

9 Prove that real roots of the equation $x^2 + 8x + k = 0$ do not exist if $k > 16$.

2 34.8°, 145.2°

8 12 units^2

10 $a^2 \geqslant 4b$

In questions 1–10, find a counter-example to disprove the assertion being made.

1 If $0 < x < 2\pi$, the only solution of the equation $\sin x = \frac{1}{2}$ is $\frac{\pi}{6}$.

2 $u > v$ and $x > y \Rightarrow ux > vy$, where u, x, v and y are real numbers.

3 x is real and $x < 4 \Rightarrow x^2 < 16$.

4 The equation $x^2 + x - a = 0$ has real roots for all real values of a.

5 The complete set of values of x for which

$$x^2 + 5x > -6$$

is $-2 < x < 3$.

6 $f(n) \equiv n^2 + n + 41$ is a prime number for all integral values of n.

7 The equation $ax^2 + bx + c = 0$ only has real roots if $b^2 > 4ac$, where a, b and c are real numbers.

8 A quadrilateral with all its interior angles equal also has all its sides equal and conversely.

9 $f(n) \equiv (n+1)(n+2)(n+3)$ is divisible by 12 for all positive integral n.

10 For all real values of x and y,

$$\sin x > \sin y \Rightarrow x > y$$

In questions 11–15, use a proof by contradiction in each case.

11 For all $x > 1$, $x + \dfrac{1}{x} > 2$.

12 Given that $x^2 < 2x$ then $0 < x < 2$.

13 Prove that there are no integers p and q such that $\dfrac{p^2}{q^2} = 2$.

14 Prove that there are an infinite number of rational numbers between 0 and 1.

15 Prove that $\sqrt{3}$ is irrational.

16 The equation $x^3 - 3x^2 - 2x + 3 = 0$ has a root in the interval $(N, N+1)$ where N is an integer. Prove that there are three

Exercise 9A

1 $\dfrac{5\pi}{6}$ is a solution.

2 There are plenty of simple counter-examples, e.g. $-5 > -7$ and $3 > 2 \nRightarrow ux > vy$.

3 Take $x \leqslant -4$ and result proposed is false.

4 For any number $a < -\frac{1}{4}$, proposition is false.

5 Solution set should be $x < -3$ or $x > -2$.

6 Breaks down for $n = 40$ when $\mathrm{f}(40) = 41^2$ for example.

7 Has real roots for $b^2 = 4ac$ too.

8 Untrue in both cases – compare with rectangle and rhombus.

9 Not true for $n = 4$ and many others.

10 Untrue in second quadrant.