Algebra – indices, surds, fractions, division, Factor theorem, partial fractions

Question 1

Simplify the following algebraic fractions.

a) $\dfrac{x^2-4x}{x^2-6x+8}$

b) $\dfrac{y^2+2y-15}{y^2-7y+12}$

c) $\dfrac{t^2+12t+36}{t^2+t-30}$

d) $\dfrac{w^2+4w-12}{w^2+9w+18}$

(a) $\dfrac{x^2-4x}{x^2-6x+8} = \dfrac{x\cancel{(x-4)}}{\cancel{(x-4)}(x-2)} = \dfrac{x}{x-2}$

(b) $\dfrac{y^2+2y-15}{y^2-7y+12} = \dfrac{(y+5)\cancel{(y-3)}}{\cancel{(y-3)}(y-4)} = \dfrac{y+5}{y-4}$

(c) $\dfrac{t^2+12t+36}{t^2+t-30} = \dfrac{\cancel{(t+6)}(t+6)}{\cancel{(t+6)}(t-5)} = \dfrac{t+6}{t-5}$

(d) $\dfrac{w^2+4w-12}{w^2+9w+18} = \dfrac{(w-2)\cancel{(w+6)}}{(w+3)\cancel{(w+6)}} = \dfrac{w-2}{w+3}$

Question 4

Simplify the following algebraic expressions giving your final answer as a single fraction in its simplest form.

a) $\dfrac{2}{x-1}-\dfrac{6}{(x-1)(2x+1)}$

b) $\dfrac{2y+5}{y+3}-\dfrac{1}{(y+3)(y+2)}$

c) $\dfrac{t+3}{(t+1)(t+2)}-\dfrac{t+1}{(t+2)(t+3)}$

d) $\dfrac{2}{w-2}+\dfrac{3w}{w^2-4}-\dfrac{5}{w+2}$

$$\boxed{\frac{4}{2x+1}},\ \boxed{\frac{2y+3}{y+2}},\ \boxed{\frac{4}{(t+1)(t+3)}},\ \boxed{\frac{14}{(w-2)(w+2)}=\frac{14}{w^2-4}}$$

(a) $\dfrac{2}{x-1}-\dfrac{6}{(x-1)(2x+1)}=\dfrac{2(2x+1)-6}{(x-1)(2x+1)}=\dfrac{4x+2-6}{(x-1)(2x+1)}=\dfrac{4x-4}{(x-1)(2x+1)}$

$=\dfrac{4\cancel{(x-1)}}{\cancel{(x-1)}(2x+1)}=\dfrac{4}{2x+1}$

(b) $\dfrac{2y+5}{y+3}-\dfrac{1}{(y+3)(y+2)}=\dfrac{(2y+5)(y+2)-1}{(y+3)(y+2)}=\dfrac{2y^2+4y+5y+10-1}{(y+3)(y+2)}$

$=\dfrac{2y^2+9y+9}{(y+3)(y+2)}=\dfrac{\cancel{(y+3)}(2y+3)}{\cancel{(y+3)}(y+2)}=\dfrac{2y+3}{y+2}$

(c) $\dfrac{t+3}{(t+1)(t+2)}-\dfrac{t+1}{(t+2)(t+3)}=\dfrac{(t+3)(t+3)-(t+1)(t+1)}{(t+1)(t+2)(t+3)}=$

$=\dfrac{\cancel{t^2}+6t+9-\cancel{t^2}-2t-1}{(t+1)(t+2)(t+3)}=\dfrac{4t+8}{(t+1)(t+2)(t+3)}$

$=\dfrac{4\cancel{(t+2)}}{(t+1)\cancel{(t+2)}(t+3)}=\dfrac{4}{(t+1)(t+3)}$

(d) $\dfrac{2}{w-2}+\dfrac{3w}{w^2-4}-\dfrac{5}{w+2}=\dfrac{2}{w-2}+\dfrac{3w}{(w-2)(w+2)}-\dfrac{5}{w+2}$

$=\dfrac{2(w+2)+3w-5(w-2)}{(w-2)(w+2)}=\dfrac{2w+4+3w-5w+10}{(w-2)(w+2)}$

$=\dfrac{14}{(w-2)(w+2)}$ or $\dfrac{14}{w^2-4}$

Question 3

Find the values of the constants in the each of the following identities.

a) $\dfrac{3x^3 - x^2 - 20x + 31}{x+3} \equiv ax^2 + bx + c + \dfrac{d}{x+3}$

b) $\dfrac{4x^3 - 7x^2 - 30x - 21}{x-4} \equiv Ax^2 + Bx + C + \dfrac{D}{x-4}$

c) $\dfrac{2x^3 - 7x + 2}{x+1} \equiv Px^2 + Qx + R + \dfrac{S}{x+1}$

$\boxed{(a,b,c,d) = (3,-10,10,1)}$, $\boxed{(A,B,C,D) = (4,9,6,3)}$, $\boxed{(P,Q,R,S) = (2,-2,-5,7)}$

Question 1

Simplify the following algebraic expressions giving your final answer as a single fraction in its simplest form:

a) $\dfrac{2x^3 + x^2}{x^2 - 4} \times \dfrac{x-2}{2x^2 - 5x - 3}$

b) $\dfrac{18y^2 - 8}{12y^2} \times \dfrac{3y^2 + 9y}{3y^2 + 11y + 6}$

c) $\dfrac{t^3 + 1}{2t^2 + 7t + 5} \times \dfrac{4t^2 - 25}{4t^2 - t + 1}$

$\boxed{\dfrac{x^2}{(x+2)(x+3)}}$, $\boxed{\dfrac{3y-2}{2y}}$ $\boxed{2t-5}$

Question 2

Simplify the following algebraic expressions giving your final answer as a single fraction in its simplest form:

a) $\dfrac{2x}{2x^2 + 3x - 5} \div \dfrac{x^3}{x^2 - x}$

b) $\dfrac{y^2 - 8y + 15}{(y-5)^2} \div \dfrac{y^2 - 9}{2y^2 + 6y}$

$\boxed{\dfrac{2}{x(2x+5)}}$, $\boxed{\dfrac{2y}{y-5}}$

Question 1

Express each of the following into partial fractions.

a) $\dfrac{6x}{(x-1)(x+2)}$

b) $\dfrac{7y-11}{(y+2)(y-3)}$

(a) $\dfrac{6x}{(x-1)(x+2)} \equiv \dfrac{A}{x-1} + \dfrac{B}{x+2}$

$6x \equiv A(x+2) + B(x-1)$

- If $x=1 \Rightarrow 6 = 3A \Rightarrow A=2$
- If $x=-2 \Rightarrow -12 = -3B \Rightarrow B=4$

$\dfrac{6x}{(x-1)(x+2)} \equiv \dfrac{2}{x-1} + \dfrac{4}{x+2}$

(b) $\dfrac{7y-11}{(y+2)(y-3)} \equiv \dfrac{A}{y+2} + \dfrac{B}{y-3}$

$7y-11 \equiv A(y-3) + B(y+2)$

- If $y=3 \Rightarrow 10 = 5B \Rightarrow B=2$
- If $y=-2 \Rightarrow -25 = -5A \Rightarrow A=5$

$\dfrac{7y-11}{(y+2)(y-3)} \equiv \dfrac{5}{y+2} + \dfrac{2}{y-3}$

Question 6

Express each of the following into partial fractions.

a) $\dfrac{x^2-4x+1}{x(x+1)(1-2x)}$

b) $\dfrac{10}{(y+1)(y+3)(2y+1)}$

(a) $\dfrac{x^2-4x+1}{x(x+1)(1-2x)} \equiv \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{1-2x}$

$x^2-4x+1 \equiv A(x+1)(1-2x) + Bx(1-2x) + Cx(x+1)$

- If $x=0$, $1=A \Rightarrow A=1$
- If $x=-1$, $6=-3B \Rightarrow B=-2$
- If $x=\frac{1}{2}$, $-\frac{3}{4}=\frac{3}{4}C \Rightarrow C=-1$

$\therefore \dfrac{x^2-4x+1}{x(x+1)(1-2x)} \equiv \dfrac{1}{x} - \dfrac{2}{x+1} - \dfrac{1}{1-2x}$ //

(b) $\dfrac{10}{(y+1)(y+3)(2y+1)} \equiv \dfrac{A}{y+1} + \dfrac{B}{y+3} + \dfrac{C}{2y+1}$

$10 \equiv A(y+3)(2y+1) + B(y+1)(2y+1) + C(y+1)(y+3)$

- If $y=-1 \Rightarrow 10=-2A \Rightarrow A=-5$
- If $y=-3 \Rightarrow 10=10B \Rightarrow B=1$
- If $y=-\frac{1}{2} \Rightarrow 10=\frac{5}{4}C \Rightarrow C=8$

$\therefore \dfrac{10}{(y+1)(y+3)(2y+1)} \equiv \dfrac{1}{y+3} - \dfrac{5}{y+1} + \dfrac{8}{2y+1}$ //

Question 7

Express each of the following into partial fractions.

a) $\dfrac{2x^2-x-3}{(x-2)(x-1)^2}$

b) $\dfrac{y^2-2y+8}{(y+2)(y-2)^2}$

(a) $\frac{2x^2-x-3}{(x-2)(x-1)^2} \equiv \frac{A}{x-2}+\frac{B}{(x-1)^2}+\frac{C}{x-1}$

$2x^2-x-3 \equiv A(x-1)^2+B(x-2)+C(x-2)(x-1)$

- If $x=1 \Rightarrow -2=-B \Rightarrow B=2$
- If $x=2 \Rightarrow 3=A \Rightarrow A=3$
- If $x=0 \Rightarrow -3=4-2B+2C$
  $-3=3-4+2C$
  $C=-1$

$\therefore \frac{2x^2-x-3}{(x-2)(x-1)^2} \equiv \frac{3}{x-2}+\frac{2}{(x-1)^2}-\frac{1}{x-1}$

(b) $\frac{y^2-2y+8}{(y+2)(y-2)^2} \equiv \frac{A}{y+2}+\frac{B}{(y-2)^2}+\frac{C}{y-2}$

$y^2-2y+8 \equiv A(y-2)^2+B(y+2)+C(y-2)(y+2)$

- If $y=2 \Rightarrow 8=4B \Rightarrow B=2$
- If $y=-2 \Rightarrow 16=16A \Rightarrow A=1$
- If $y=0 \Rightarrow 8=4A+2B-4C$
  $8=4+4-4C$
  $C=0$

$\frac{y^2-2y+8}{(y+2)(y-2)^2} \equiv \frac{1}{y+2}+\frac{2}{(y-2)^2}$

(c) $\frac{-3t^2+12t+7}{(t-3)(t+1)^2} \equiv \frac{A}{t-3}+\frac{B}{(t+1)^2}+\frac{C}{t+1}$

$-3t^2+12t+7 \equiv A(t+1)^2+B(t-3)+C(t+1)(t-3)$

- $t=3$ $\quad 16=16A \Rightarrow A=1$
- $t=-1$ $\quad -8=-4B \Rightarrow B=2$
- $t=0$ $\quad 7=A-3B-3C$
  $7=1-6-3C$
  $C=-4$

$\therefore \frac{-3t^2+12t+7}{(t-3)(t+1)^2} \equiv \frac{1}{t-3}+\frac{2}{(t+1)^2}-\frac{4}{t+1}$

(d) $\frac{-3w^2+10w-11}{(w-2)(w-1)^2} \equiv \frac{A}{w-2}+\frac{B}{(w-1)^2}+\frac{C}{w-1}$

$-3w^2+10w-11 \equiv A(w-1)^2+B(w-2)+C(w-1)(w-2)$

- If $w=1$ $\quad -4=-B \Rightarrow B=4$
- If $w=2$ $\quad -3=A \Rightarrow A=-3$
- If $w=0$ $\quad -11=A-2B+2C$
  $-11=-3-8+2C$
  $C=0$

$\therefore \frac{-3w^2+10w-11}{(w-2)(w-1)^2}=\frac{4}{(w-1)^2}-\frac{3}{w-2}$

Indicies

Question 14 (***)

$$6x^{-\frac{1}{2}}-x^{\frac{1}{2}}=5.$$

a) Show that the substitution $y=x^{\frac{1}{2}}$ transforms the above indicial equation into the quadratic equation

$$y^2+5y-6=0.$$

b) Solve the quadratic equation and hence find the root of the **indicial** equation.

$\boxed{x=1}$

a) $6x^{-\frac{1}{2}}-x^{\frac{1}{2}}=5$

$\Rightarrow \frac{6}{x^{\frac{1}{2}}}-x^{\frac{1}{2}}=5$

LET $y=x^{\frac{1}{2}}$

$\Rightarrow \frac{6}{y}-y=5$

$\Rightarrow 6-y^2=5y$

$\Rightarrow 0=y^2+5y-6$

AS REQUIRED

b) $(y+6)(y-1)=0$

$y = 1$ or -6

$x^{\frac{1}{2}}=\sqrt{x}= 1$ or $\cancel{-6}$

$x=1$

Question 15 (***)

The points $(2,14)$ and $(6,126)$ lie on the curve with equation

$$y = ax^n, \; x \in \mathbb{R}$$

where a and n are non zero constants.

Find the value of a and the value of n.

$\boxed{a = \frac{7}{2}}$, $\boxed{n = 2}$

$y = ax^n$

$(2,14) \Rightarrow 14 = a \times 2^n$

$(6,126) \Rightarrow 126 = a \times 6^n$ $\Big\}$ $\Rightarrow$ DIVIDE EQUATIONS SIDE BY SIDE

$$\frac{\cancel{a} \times 6^n}{\cancel{a} \times 2^n} = \frac{126}{14}$$

$$\frac{6^n}{2^n} = 9$$

$$\left(\frac{6}{2}\right)^n = 9$$

$$3^n = 9$$

$$n = 2$$

AND

$$14 = a \times 2^n$$

$$14 = a \times 2^2$$

$$14 = 4a$$

$$7 = 2a$$

$$a = \frac{7}{2}$$

Question 30 (***+)

Solve the following simultaneous equations without using a calculator

$$8^y = 4^{2x+1}$$

$$27^{2y} = 9^{x-3}$$

$\boxed{\left(-\frac{5}{3}, -\frac{14}{9}\right)}$

$$\left.\begin{array}{l} 8^y = 4^{2x+1} \\ 27^{2y} = 9^{x-3} \end{array}\right\} \Rightarrow \left.\begin{array}{l} (2^3)^y = (2^2)^{2x+1} \\ (3^3)^{2y} = (3^2)^{x-3} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2^{3y} = 2^{4x+2} \\ 3^{6y} = 3^{2x-6} \end{array}\right\} \Rightarrow$$

$$\left.\begin{array}{l} 3y = 4x+2 \\ 6y = 2x-6 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 6y = 8x+4 \\ 6y = 2x-6 \end{array}\right\} \Rightarrow 8x+4 = 2x-6$$

$$\Rightarrow 6x = -10$$

$$\Rightarrow x = -\frac{5}{3}$$

$$\therefore 3y = 4x+2$$

$$\Rightarrow 3y = 4\left(-\frac{5}{3}\right)+2$$

$$\Rightarrow 3y = -\frac{20}{3}+2$$

$$\Rightarrow 3y = -\frac{20}{3}+\frac{6}{3}$$

$$\Rightarrow 3y = \frac{-14}{3}$$

$$\Rightarrow y = -\frac{14}{9}$$

Surds

Question 49 (***+)

$$p = \frac{3}{2},\ q = \frac{9-\sqrt{17}}{4} \quad \text{and} \quad r = \frac{9+\sqrt{17}}{4}.$$

Prove that

$$p+q+r = pqr.$$

$\boxed{}$, $\boxed{\text{proof}}$

- $p+q+r = \frac{3}{2} + \frac{9-\sqrt{17}}{4} + \frac{9+\sqrt{17}}{4} = \frac{6+9-\cancel{\sqrt{17}}+9+\cancel{\sqrt{17}}}{\cancel{4}} = 6$
- $pqr = \left(\frac{3}{2}\right)\left(\frac{9-\sqrt{17}}{4}\right)\left(\frac{9+\sqrt{17}}{4}\right) = \frac{3(9-\sqrt{17})(9+\sqrt{17})}{32}$

$$= \frac{3(81+\cancel{9\sqrt{17}}-\cancel{9\sqrt{17}}-17)}{32} = \frac{3\times\cancel{64}^{2}}{\cancel{32}_{1}} = 6$$

$$\therefore p+q+r = 6 = pqr$$

AS REQUIRED

Question 5 (**+)

Show clearly that

$$1+\frac{x-8}{x^2+2x-8}-\frac{2}{x+4}\equiv\frac{x-p}{x-q},$$

stating the value of each of the integer constants, p and q.

$\boxed{}$, $\boxed{p=3}$, $\boxed{q=2}$

$$1+\frac{x-8}{x^2+2x-8}-\frac{2}{x+4}=\frac{1}{1}+\frac{x-8}{(x+4)(x-2)}-\frac{2}{x+4}$$

$$=\frac{1(x+4)(x-2)+(x-8)-2(x-2)}{(x+4)(x-2)}=\frac{x^2+4x-2x-8+x-8-2x+4}{(x+4)(x-2)}$$

$$=\frac{x^2+x-12}{(x+4)(x-2)}=\frac{(x+4)(x-3)}{(x+4)(x-2)}=\frac{x-3}{x-2}$$ AS REQUIRED

Question 9 (***)

Solve the equation

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1,\ x\neq3,\ x\neq7.$$

$\boxed{x=-8,6}$

$$\frac{2}{x-3}+\frac{13}{x^2+4x-21}=1$$

$$\Rightarrow \frac{2}{x-3}+\frac{13}{(x-3)(x+7)}=1$$

$$\Rightarrow \frac{2(x+7)+13}{(x-3)(x+7)}=1$$

$$\Rightarrow \frac{2x+14+13}{x^2+4x-21}=1$$

$$\Rightarrow 2x+27=x^2+4x-21$$

$$\Rightarrow 0=x^2+2x-48$$

$$\Rightarrow (x+8)(x-6)=0$$

$$\Rightarrow x=\begin{cases}-8\\6\end{cases}$$

Factor Theorem

Question 5 (***)

A cubic polynomial is defined as

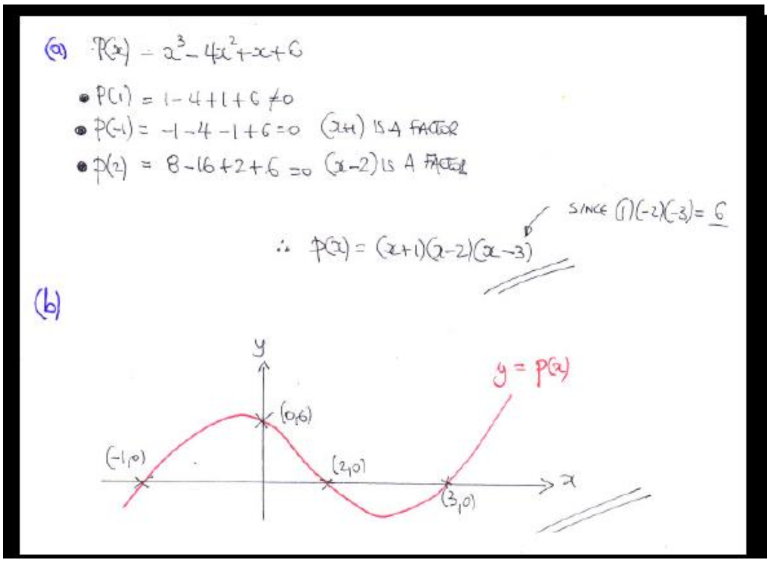
$$p(x) \equiv x^3 - 4x^2 + x + 6, \; x \in \mathbb{R}.$$

a) By considering the factors of 6, or otherwise, express $p(x)$ as the product of three linear factors.

b) Sketch the graph of $p(x)$.

The sketch must include the coordinates of any points where the graph of $p(x)$ meets the coordinate axes.

$$\boxed{p(x) = (x-3)(x-2)(x+1)}$$



Question 22 (***)

$$f(x) \equiv 4x^3 + 9x^2 + 3x + 2$$

a) Use the factor theorem to show that $(x+2)$ is a factor of $f(x)$.

b) Given further that

$$f(x) \equiv (x+2)\left(ax^2 + bx + c\right),$$

find the value of each of the constants a, b and c.

c) Show that the equation $f(x) = 0$ has only one real root.

$$\boxed{a = 4}, \; \boxed{b = 1}, \; \boxed{c = 1}$$

(a) $f(x) = 4x^3+9x^2+3x+2$

$f(-2) = 4(-2)^3+9(-2)^2+3(-2)+2 = -32+36-6+2 = 38-38 = 0$

$\therefore$ $(x+2)$ IS A FACTOR

(b)

$$\begin{array}{r} 4x^2+x+1 \\ x+2\,\overline{)\,4x^3+9x^2+3x+2} \\ -4x^3-8x^2 \\ \hline x^2+3x+2 \\ -x^2-2x \\ \hline x+2 \\ -x-2 \\ \hline 0 \end{array}$$

$\therefore f(x) = (x+2)(4x^2+x+1)$

IE $a=4$
$b=1$
$c=1$

(c)

- $x=-2$ IS ONE OF THE SOLUTIONS
- CHECK THE DISCRIMINANT OF THE QUADRATIC TERM $4x^2+x+1$

$b^2-4ac = 1^2-4\times4\times1 = -15 < 0$

$\therefore$ NO ROOTS

- HENCE $x=-2$ IS THE ONLY ROOT

Quadratics and inequalities

Question 9 (**+)

$$f(x) = x^2+4x+12,\ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $(x+a)^2+b$, where a and b are integers.

b) Determine the greatest value of $\dfrac{1}{f(x)}$.

$\boxed{}$, $\boxed{a=2}$, $\boxed{b=8}$, $\boxed{\frac{1}{8}}$

(a)

$f(x) = x^2+4x+12$

$f(x) = (x+2)^2-2^2+12$

$f(x) = (x+2)^2-4+12$

$f(x) = (x+2)^2+8$

IE $a=2$
$b=8$

(b)

$\dfrac{1}{f(x)} = \dfrac{1}{x^2+4x+12}$

$\dfrac{1}{f(x)} = \dfrac{1}{(x+2)^2+8}$

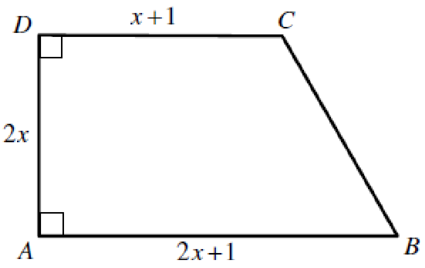
TO BE AS LARGE AS POSSIBLE WE NEED THE SMALLEST POSSIBLE DENOMINATOR

BUT SMALLEST VALUE OF $(x+2)^2+8$ IS 8

$\therefore \dfrac{1}{f(x)}_{MAX} = \dfrac{1}{8}$

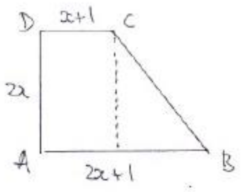
Question 23 (***)

A right angled trapezium $ABCD$ is shown in the figure below.



The trapezium has parallel sides AB and CD of lengths $(2x+1)$cm and $(x+1)$cm. The height of the trapezium AD is $2x$ cm.

Given that the area of the trapezium is 16 cm^2, determine the exact length of BC.



$$\frac{(x+1)+(2x+1)}{2} \times 2x = 16$$

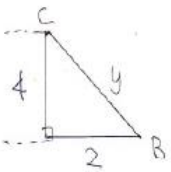
$$\frac{3x+2}{\cancel{2}} \times \cancel{2}x = 16$$

$$(3x+2)x = 16$$

$$3x^2+2x-16=0$$

$$(3x+8)(x-2)=0$$

$$x = 2, \ \cancel{-\tfrac{8}{3}}$$



BY PYTHAGORAS $y^2 = 4^2+2^2$

$y^2 = 20$

$y = \sqrt{20} = 2\sqrt{5}$

Question 33 (***+)

$$f(x) = 9x^2 + 18x - 7, \ x \in \mathbb{R}.$$

a) Solve the equation $f(x) = 0$.

b) Express $f(x)$ in the form

$$f(x) = 9(x + A)^2 + B,$$

where A and B are integer constants.

c) State the minimum value of $f(x)$.

d) Sketch the graph of $f(x)$, indicating clearly the coordinates of the points where the graph of $f(x)$ meets the coordinate axes.

$\boxed{}$, $\boxed{x = -\frac{7}{3}, \frac{1}{3}}$, $\boxed{A = 1}$, $\boxed{B = -16}$, $\boxed{f(x)_{\text{min}} = -16}$

(a) $f(x) = 9x^2 + 18x - 7$

$f(x) = 0$

$9x^2 + 18x - 7 = 0$

$(3x + 7)(3x - 1) = 0$

$x = \frac{1}{3}, -\frac{7}{3}$

(b) $f(x) = 9\left[x^2 + 2x - \frac{7}{9}\right]$

$f(x) = 9\left[(x+1)^2 - 1 - \frac{7}{9}\right]$

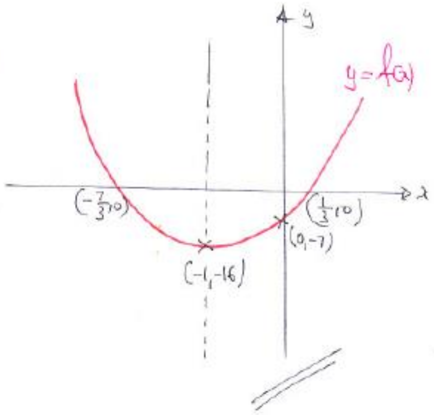
$f(x) = 9\left[(x+1)^2 - \frac{16}{9}\right]$

$f(x) = 9(x+1)^2 - 16$

(c) MINIMUM VALUE OF $f(x)$ IS -16

(IT OCCURS WHEN $x = -1$)

(d)



Inequalities

Question 7 (***+)

Show that the quadratic equation

$$(k+1)x^2+2kx+k=1$$

has two distinct real roots for all values of k, except for one value which must be stated.

$\boxed{k \neq -1}$

$(k+1)x^2+2kx+k=1$

$(k+1)x^2+2kx+(k-1)=0$

$b^2-4ac=(2k)^2-4(k+1)(k-1)$

$=4k^2-4(k^2-1)$

$=4k^2-4k^2+4$

$=4$

>0

$\therefore$ ALWAYS TWO DISTINCT ROOTS UNLESS $k=-1$, BECAUSE QUADRATIC IS LINEAR!

Question 8 (***+)

Find the range of values of the constant m so that the equation

$$x^2+(m+2)x+3m=2,$$

has two distinct roots.

$\boxed{m<2 \quad \text{or} \quad m>6}$

$x^2+(m+2)x+3m=2$

$x^2+(m+2)x+(3m-2)=0$

TWO DISTINCT ROOTS $\Rightarrow b^2-4ac>0$

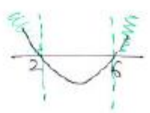
$\Rightarrow (m+2)^2-4\times1\times(3m-2)>0$

$\Rightarrow m^2+4m+4-12m+8>0$

$\Rightarrow m^2-8m+12>0$

$\Rightarrow (m-2)(m-6)>0$

$\Rightarrow$ C.V $= \begin{cases}2\\6\end{cases}$



$m<2$ or $m>6$

Question 26 (****)

The straight line with equation

$$y = k(4x - 17),$$

does **not** intersect with the quadratic with equation

$$y = 13 - 8x - x^2.$$

a) Show clearly that

$$4k^2 + 33k + 29 < 0.$$

b) Hence find the range of possible values of k.

$$\boxed{-\frac{29}{4} < k < -1}$$

(a) $y = k(4x-17)$

$y = 13 - 8x - x^2$

$\Rightarrow k(4x-17) = 13 - 8x - x^2$

$\Rightarrow x^2 + 8x - 13 + 4kx - 17k = 0$

$\Rightarrow x^2 + (8+4k)x + (-13-17k) = 0$

NO INTERSECTIONS $\Rightarrow$ NO ROOTS

$b^2 - 4ac < 0$

$\Rightarrow (8+4k)^2 - 4 \times 1 \times (-13-17k) < 0$

$\Rightarrow 64 + 64k + 16k^2 + 4(13+17k) < 0 \quad (\div 4)$

$\Rightarrow 16 + 16k + 4k^2 + 13 + 17k < 0$

$4k^2 + 33k + 29 < 0$

AS REQUIRED

(b) FACTORIZING

$(k+1)(4k+29) < 0$

$C.V = \begin{cases} -1 \\ -\frac{29}{4} \end{cases}$

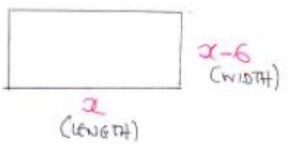
$\therefore -\frac{29}{4} < k < -1$

Question 5 (***)

A rectangle is such so that its length is 6 cm greater than its width.

Given the area of the rectangle is at least 40 cm^2, determine the range of the possible values of the **length** of the rectangle.

$\boxed{}$, $\boxed{\text{length} > 10}$



$x(x-6) > 40$

$x^2 - 6x - 40 > 0$

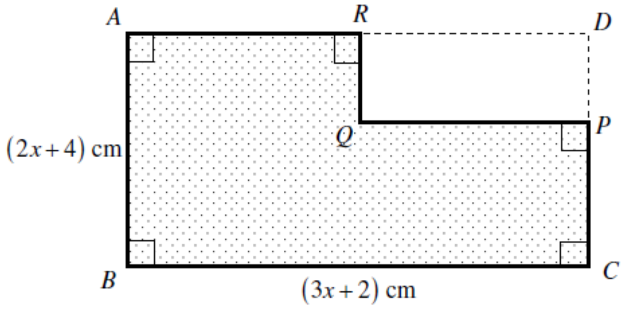
$(x+4)(x-10) > 0$

$C.V = \begin{cases} 10 \\ -4 \end{cases}$

$x < -4$ OR $x > 10$

BUT IN THIS CONTEXT $x > 10$

Question 10 (***+)



A rectangle $ABCD$ measures $(3x+2)$ cm by $(2x+4)$ cm.

A second rectangle $PQRD$ is removed from the rectangle $ABCD$, as shown in the figure above. The perimeter of the composite shape $ABCPRQ$ is greater than 27 cm but less than 52 cm.

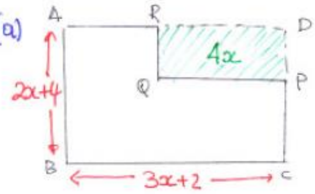
a) Find the range of the possible values of x.

The area of the rectangle $PQRD$ is $4x$ cm^2.

b) Given further that the area of the composite shape $ABCPRQ$ is less than 98 cm^2, determine an amended range of the possible values of x.

$\boxed{}$, $\boxed{1.5 < x < 4}$, $\boxed{1.5 < x < 3}$

(a)



$27 < P < 52$

$\Rightarrow 27 < 2[(3x+2)+(2x+4)] < 52$

$\Rightarrow 27 < 2(5x+6) < 52$

$\Rightarrow 27 < 10x+12 < 52$

$\Rightarrow 15 < 10x < 40$

$\Rightarrow 1.5 < x < 4$

(b) $A < 98$

$\Rightarrow (2x+4)(3x+2) - 4x < 98$

$\Rightarrow 6x^2 + 16x + 8 - 4x < 98$

$\Rightarrow 6x^2 + 12x - 90 < 0$

$\Rightarrow x^2 + 2x - 15 < 0$

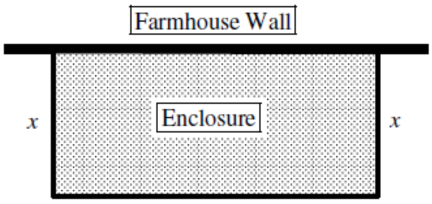
$\Rightarrow (x-3)(x+5) < 0$

$-5 < x < 3$

$\therefore$ As $x > 0$

$1.5 < x < 3$

Question 15 (***+)



The figure above shows the plan of a rectangular enclosure to be built next to a farmhouse. One of the farmhouse's walls will form one of the sides of the enclosure and 25 metres of fencing will form the other three sides.

The width of the enclosure is x metres, as shown in the figure.

The area of the enclosure must be at most 75 m^2.

Given further that the width of the enclosure must be at least 3 metres but no more than 9 metres, determine the range of the possible values of x.

$\boxed{}$, $\boxed{3 < x < 5 \cup 7.5 < x < 9}$

25 metres of fencing available

AREA < 75

$\Rightarrow x(25-2x) < 75$

$\Rightarrow 25x - 2x^2 < 75$

$\Rightarrow -2x^2 + 25x - 75 < 0$

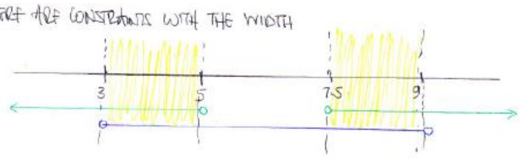
$\Rightarrow 2x^2 - 25x + 75 > 0$

$\Rightarrow (2x-15)(x-5) > 0$

$\text{C.V} = \begin{cases} 5 \\ \frac{15}{2} \end{cases}$

$x < 5$ or $x > 7.5$

BUT THERE ARE CONSTRAINTS WITH THE WIDTH



$\therefore$ $3 < x < 5$ OR $7.5 < x < 9$

Graphs – transformations, modulus

Question 4 (***)

The curve C has equation

$$y = x^3 - 9x .$$

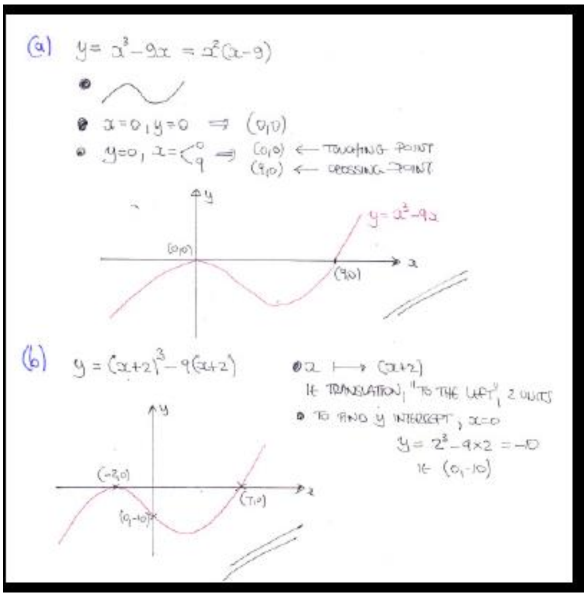
a) Sketch the graph of C.

b) Hence sketch on a separate diagram the graph of

$$y = (x+2)^3 - 9(x+2).$$

Each of the two sketches must include the coordinates of all the points where the curve meets the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 10 (*+)**

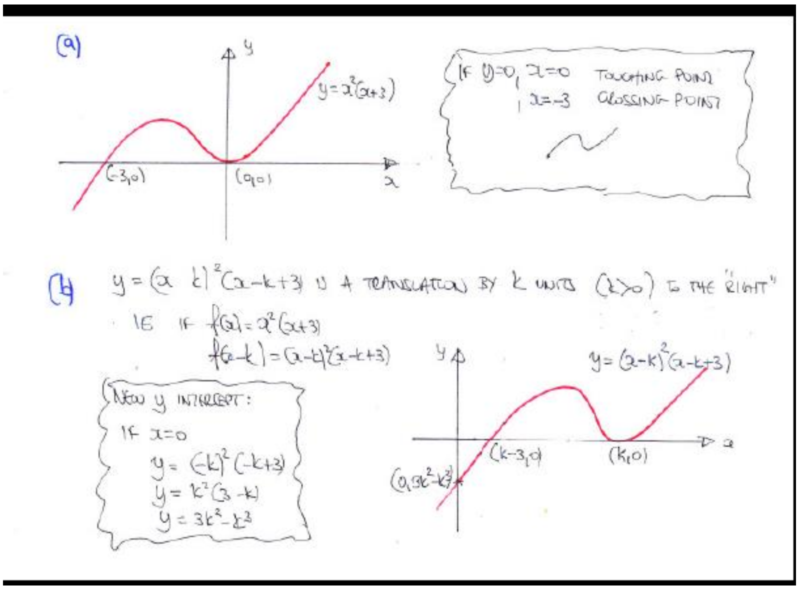
Sketch on separate diagrams the curve with equation …

a) … $y = x^2(x+3)$.

b) … $y = (x-k)^2(x-k+3)$, where k is a constant such that $k > 3$.

Both sketches must include the coordinates, in terms of k where appropriate, of any points where each of the curves meets the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 3 (***)

$$f(x) = \sqrt{x}, \; x \in \mathbb{R}, \; x \geq 0 .$$

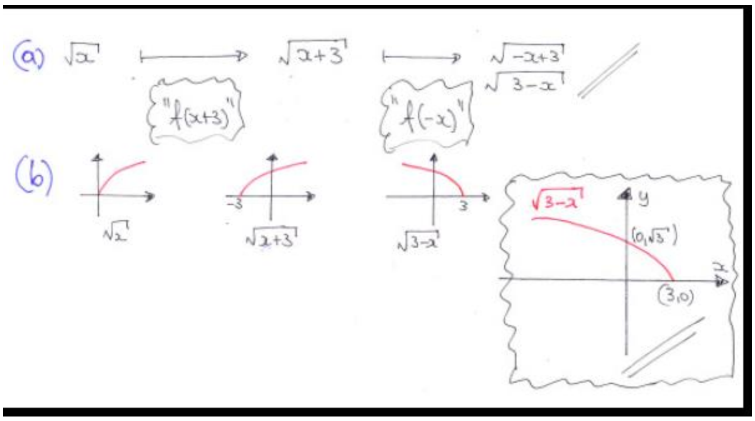
The graph of $f(x)$ is translated by 3 units in the negative x direction, followed by a reflection in the y axis, forming the graph of $g(x)$.

a) Find the equation of $g(x)$.

b) Sketch the graph of $g(x)$.

The sketch must include the coordinates of all the points where the curve meets the coordinate axes.

$$\boxed{g(x) = \sqrt{3-x}}$$



Question 8 (***+)

The curve C has equation

$$y = \frac{2x+3}{x-2}, \ x \in \mathbb{R}, \ x \neq 2.$$

a) Show clearly that

$$\frac{2x+3}{x-2} \equiv 2 + \frac{7}{x-2}.$$

b) Find the coordinates of the points where C meets the coordinate axes.

c) Sketch the graph of C showing clearly the equations of any asymptotes.

$$\boxed{\left(0,-\tfrac{3}{2}\right),\left(-\tfrac{3}{2},0\right)}$$

(a) $y = \frac{2x+3}{x-2}$ $\quad y = \frac{2(x-2)+7}{(x-2)} = \frac{2(x-2)}{x-2} + \frac{7}{x-2} = 2 + \frac{7}{x-2}$

OR "USING THE ANSWER" $y = 2 + \frac{7}{x-2} = \frac{2(x-2)}{(x-2)} + \frac{7}{x-2} = \frac{2x-4}{x-2} + \frac{7}{x-2}$

$= \frac{2x+3}{x-2}$

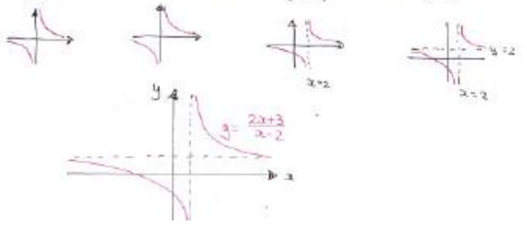
(b) • WHEN $x=0$, $y = \frac{3}{-2}$ $\Rightarrow$ $\left(0,-\frac{3}{2}\right)$

• WHEN $y=0$, $0 = \frac{2x+3}{x-2}$

$2x+3=0$

$x=-\frac{3}{2}$ $\Rightarrow$ $\left(-\frac{3}{2},0\right)$

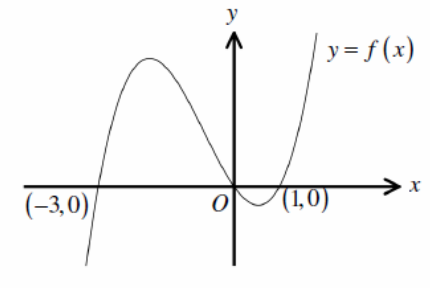
(c) $\frac{1}{x} \longmapsto 7\left(\frac{1}{x}\right) \longmapsto \frac{7}{(x-2)} \longmapsto \frac{7}{x-2}+2$



Question 5 (**)

The figure below shows the graph of the curve with equation $y = f(x)$.

The curve meets the x axis at $(-3,0)$, at $(1,0)$ and at the origin O.



Sketch on separate diagrams the graph of …

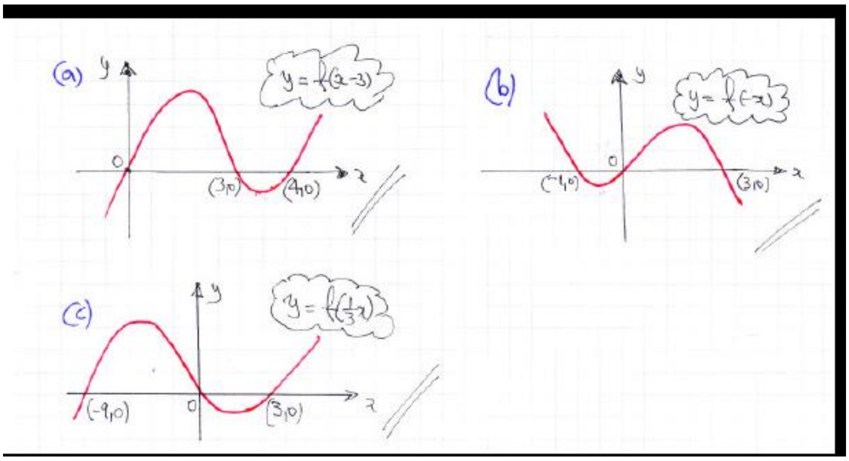
a) … $y = f(x-3)$.

b) … $y = f(-x)$.

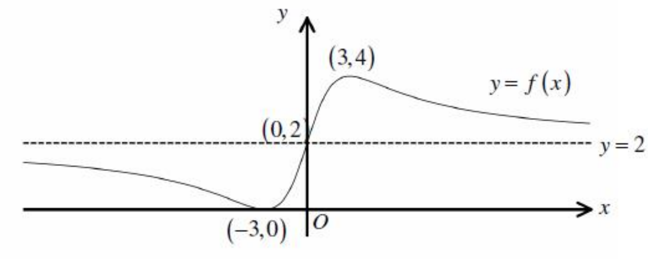
c) … $y = f\left(\frac{1}{3}x\right)$.

Each sketch must include the coordinates of any points where the graph meets the coordinate axes.

$\boxed{}$, $\boxed{\text{graph}}$



Question 17 (**+)



The figure above shows the graph of a curve with equation $y = f(x)$. The curve meets the x axis at $(-3,0)$ and the y axis at $(0,2)$. The curve has a maximum at $(3,4)$ and a minimum at $(-3,0)$.

The line with equation $y = 2$ is a horizontal asymptote to the curve.

Sketch on separate diagrams the graph of …

a) … $y = f(x+3)$.

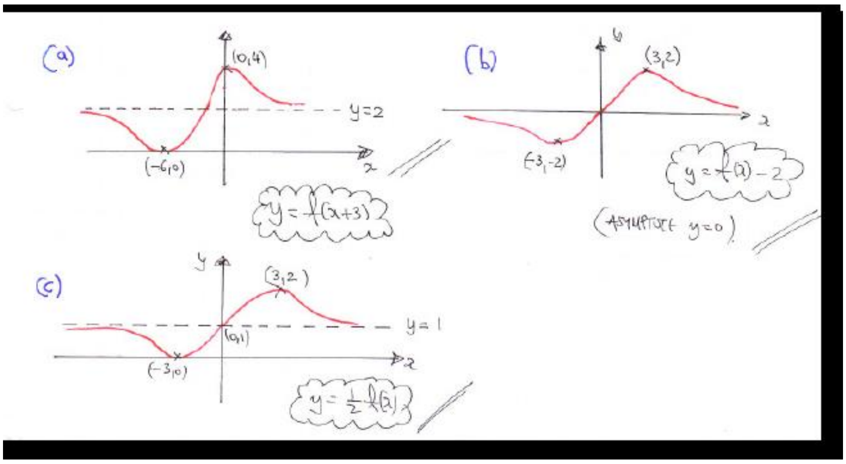
b) … $y = f(x) - 2$.

c) … $y = \frac{1}{2} f(x)$.

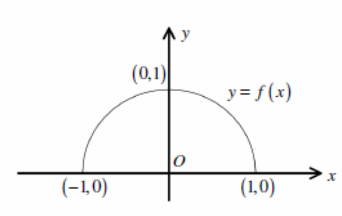
Each of the sketches must include

- the coordinates of any points where the graph meets the coordinate axes.
- the coordinates of any minimum or maximum points of the curve.
- any asymptotes to the curve, clearly labelled.

[ ], graph



Question 32 (***+)



The figure above shows the graph of a function with equation $y = f(x)$.

The curve meets has a maximum point on the y axis at the point with coordinates $(0,1)$.

It meets the x axis at the points with coordinates $(-1,0)$ and $(1,0)$.

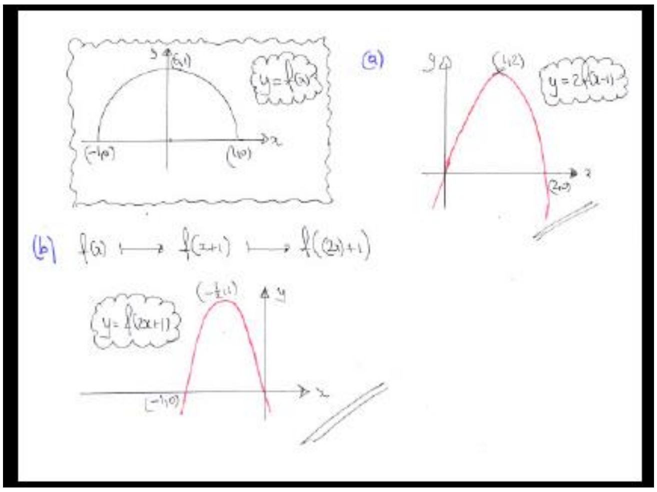
Sketch on separate diagrams the graphs of …

a) … $y = 2f(x-1)$.

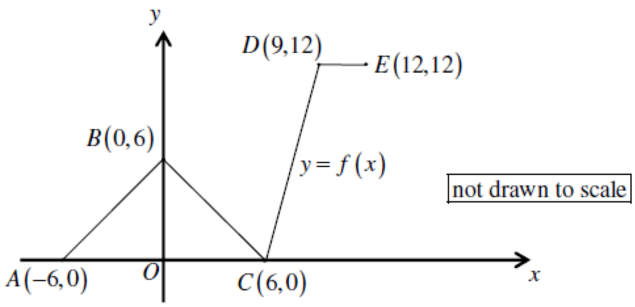
b) … $y = f(2x-1)$.

The sketches must include the coordinates of any points where the transformed graph meets the coordinate axes, and the coordinates of its maximum point.

graph



Question 45 (****)



The figure above shows the graph of a function with equation $y = f(x)$.

The graph consists of four straight line segments joining the points $A(-6,0)$, $B(0,6)$, $C(6,0)$, $D(9,12)$ and $E(12,12)$.

a) Write down, with some justification, the number of roots of the equation …

i. … $f(x) = 2$.

ii. … $f(x) = x$

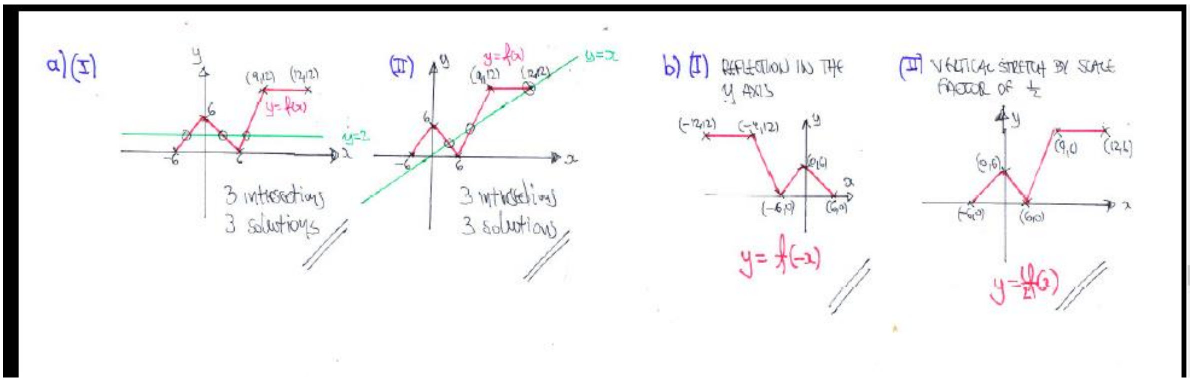
b) Sketch on separate diagrams the graph of …

i. … $y = f(-x)$.

ii. … $y = \frac{1}{2} f(x)$.

Each sketch must include the new coordinates of A, B, C, D and E.

$\boxed{}$, $\boxed{\text{3 roots}}$, $\boxed{\text{3 roots}}$, $\boxed{\text{graph}}$



Question 10 (***)

The functions f and g are defined as

$$f(x) = |2x-4|, \quad x \in \mathbb{R}$$

$$g(x) = |x|, \quad x \in \mathbb{R}.$$

a) Sketch in the same diagram the graph of f and the graph of g.

Mark clearly in the sketch the coordinates of any x or y intercepts.

b) Solve the equation

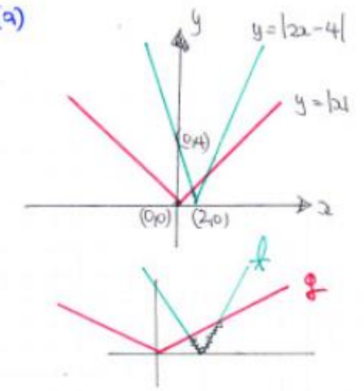
$$f(x) = g(x).$$

c) Hence, or otherwise, solve the inequality

$$f(x) < g(x).$$

$\boxed{(0,0),(2,0),(0,4)}$, $\boxed{x = \frac{4}{3}, 4}$, $\boxed{\frac{4}{3} < x < 4}$

(a)



(b)

$$\begin{cases} 2x-4 = x \\ 2x-4 = -x \end{cases}$$

$$\begin{cases} x = 4 \\ 3x = 4 \end{cases}$$

$$\therefore x = \begin{cases} 4 \\ \frac{4}{3} \end{cases}$$

FROM SKETCH

$$\frac{4}{3} < x < 4$$

Question 15 (***)

The curve C_1 and the curve C_2 have respective equations

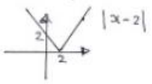
$$y = |x| \quad \text{and} \quad y = |x-2|+1.$$

a) Sketch the graph of C_2, indicating the coordinates of any intercepts with the coordinate axes.

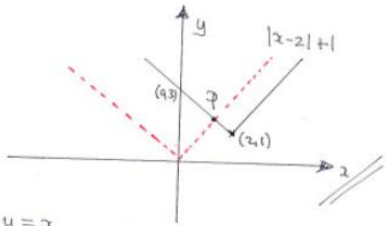
b) Determine the coordinates of the point of intersection between the graph of C_1 and the graph of C_2.

$\boxed{\left(\frac{3}{2}, \frac{3}{2}\right)}$

(a)



So



(b)

$$\left.\begin{array}{l} y = x \\ y = 1-(x-2) \end{array}\right\} \Rightarrow \left.\begin{array}{l} y = x \\ y = 3-x \end{array}\right\} \Rightarrow$$

$\Rightarrow x = 3 - x$

$\Rightarrow 2x = 3$

$\Rightarrow x = \frac{3}{2}$

$\therefore P\left(\frac{3}{2}, \frac{3}{2}\right)$

Question 36 (***+)

$$f(x) \equiv \ln x, \; x \in \mathbb{R}, \; x > 0$$

$$g(x) \equiv 2\ln(x+\mathrm{e}), \; x \in \mathbb{R}, \; x > -\mathrm{e}.$$

a) Describe mathematically the transformations which map the graph of $f(x)$ onto the graph of $g(x)$.

b) Sketch the graph of $y = |g(x)|$, indicating the coordinates of any intercepts of the graph with the coordinate axes.

c) Solve the equation

$$|g(x)| = 2.$$

d) Hence solve the inequality $|g(x)| \geq 2$.

$\boxed{x = 0, \; \mathrm{e}^{-1} - \mathrm{e}}$, $\boxed{-\mathrm{e} < x \leq \mathrm{e}^{-1} - \mathrm{e} \;\text{ or }\; x \geq 0}$

(a) $y=\ln x \longmapsto y=2\ln x \longmapsto y=2\ln(x+e)$

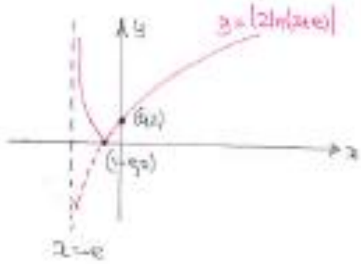
VERTICAL STRETCH BY SCALE FACTOR 2

TRANSLATION, TO THE "LEFT" BY e UNITS

OR

TRANSLATION BY THE VECTOR $\begin{pmatrix}-e\\0\end{pmatrix}$

(b)



$2\ln(x+e)=0$

$\ln(x+e)=0$

$x+e=e^0$

$x+e=1$

$x=1-e$

(c) $|2\ln(x+e)|=2$

$\begin{cases}2\ln(x+e)=2\\2\ln(x+e)=-2\end{cases} \Rightarrow$

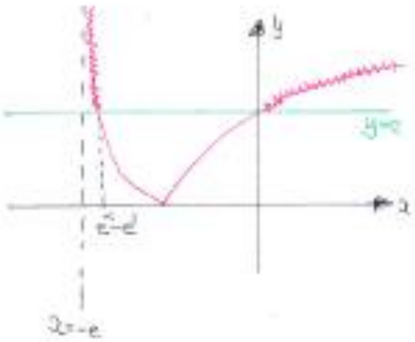
$\begin{cases}\ln(x+e)=1\\\ln(x+e)=-1\end{cases} \Rightarrow$

$\begin{cases}x+e=e^1\\x+e=e^{-1}\end{cases} \Rightarrow$

$x=0$

$x=e^{-1}-e$

(d)



FROM GRAPH $-e < x \leqslant e^{-1}-e$

OR

$x \geqslant 0$

Question 57 (****)

The functions f and g are defined as

$$f(x) = |x| - a, \; x \in \mathbb{R},$$

$$g(x) = |2x + 4a|, \; x \in \mathbb{R},$$

where a is a positive constant.

a) Sketch in the same diagram the graph of $f(x)$ and the graph of $g(x)$.

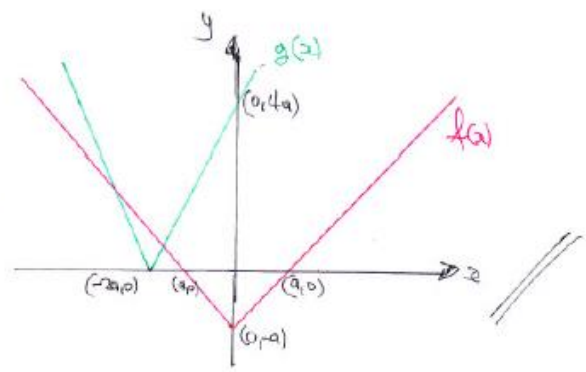
The sketch must include the coordinates of any points where the graphs meet the coordinate axes.

b) Find the solutions of the equation

$$|x| - 3 = |2x + 12|.$$

$\boxed{}$, $\boxed{x = -5, \; x = -9}$

(a)



(b)

$|x| - a = |2x + 4a|$ LET $a = 3$

$|x| - 3 = |2x + 12|$

$-x - 3 = 2x + 12$

$-x - 3 = -2x - 12$ FROM GRAPH

$\Rightarrow \begin{cases} 3x = -15 \\ x = -9 \end{cases}$

$x = -5, \; -9$

Coordinate geometry – lines, circles

Question 18 **(**+)**

The points A and B have coordinates $(2,3)$ and $(6,1)$, respectively.

a) Find an equation of the straight line l_1 which passes through A and B.

The line l_2 has gradient $\frac{1}{4}$ and meets the y axis at the point $\left(0,\frac{13}{4}\right)$.

The two lines, l_1 and l_2, intersect at the point C.

b) Show clearly that the length of AC is exactly $\frac{1}{2}\sqrt{5}$.

$\boxed{2y+x=8}$

(a) GRADIENT $= \frac{y_2-y_1}{x_2-x_1} = \frac{1-3}{6-2} = \frac{-2}{4} = -\frac{1}{2}$

USING $(2,3)$ $\Rightarrow y-y_0 = m(x-x_0)$

$\Rightarrow y-3 = -\frac{1}{2}(x-2)$

$\Rightarrow 2y-6 = -x+2$

$\Rightarrow 2y+x=8$

(b) LINE l_2: $y=\frac{1}{4}x+\frac{13}{4}$ BY INSPECTION

l_1: $2y+x=8$

THUS $2\left(\frac{1}{4}x+\frac{13}{4}\right)+x=8$

$\frac{1}{2}x+\frac{13}{2}+x=8$

$x+13+2x=16$

$3x=3$

$\boxed{x=1}$

$\therefore y=\frac{1}{4}\times 1+\frac{13}{4}=\frac{14}{4}=\frac{7}{2}$

$\boxed{y=\frac{7}{2}}$

$A(2,3)$ & $C\left(1,\frac{7}{2}\right)$

USING $d=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$|AC|=\sqrt{\left(\frac{7}{2}-3\right)^2+(1-2)^2}$

$|AC|=\sqrt{\left(\frac{1}{2}\right)^2+(-1)^2}$

$|AC|=\sqrt{\frac{1}{4}+1}=\sqrt{\frac{5}{4}}$

$|AC|=\frac{\sqrt{5}}{\sqrt{4}}=\frac{\sqrt{5}}{2}$

AS REQUIRED

Question 24 (***)

The straight line L_1 passes through the points $A(13,5)$ and $B(9,2)$.

a) Find an equation for L_1.

The point D lies on L_1 and the point C has coordinates $(2,3)$.

The straight line L_2 passes through C and D, and is perpendicular to L_1.

b) Determine an equation for L_2, giving the answer in the form $ax+by=c$, where a, b and c are integers.

c) Find the coordinates of D.

$\boxed{4y=3x-19}$, $\boxed{3y+4x=17}$, $\boxed{D(5,-1)}$

(a) GRADIENT $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{2-5}{9-13} = \frac{-3}{-4} = \frac{3}{4}$

$m = \frac{3}{4}$ $A(13,5)$

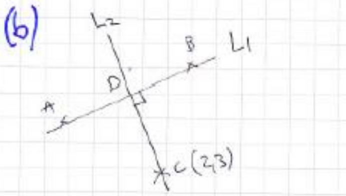
$y - y_0 = m(x - x_0)$

$y - 5 = \frac{3}{4}(x - 13)$

$4y - 20 = 3x - 39$

$4y = 3x - 19$ //

(b)



GRADIENT $L_2 = -\frac{4}{3}$ & $C(2,3)$

Thus $y - y_0 = m(x - x_0)$

$y - 3 = -\frac{4}{3}(x - 2)$

$3y - 9 = -4x + 8$

$4x + 3y = 17$ //

(c) $L_1 \Rightarrow 4y - 3x = -19$ $(\times 4)$

$L_2 \Rightarrow 3y + 4x = 17$ $(\times 3)$

$16y - 12x = -76$

$9y + 12x = 51$

$25y = -25$

$y = -1$

AND $3y + 4x = 17$

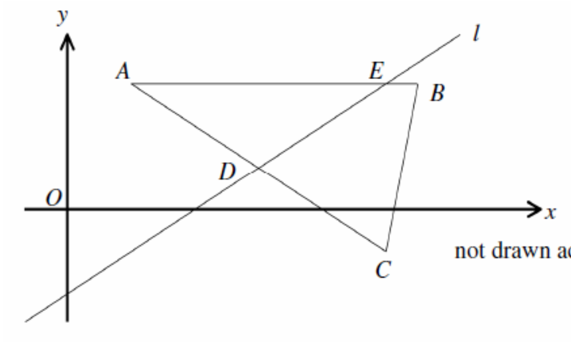
$-3 + 4x = 17$

$4x = 20$

$x = 5$ $\therefore D(5,-1)$ //

Question 29 (***)

.



The figure above shows a triangle with vertices at $A(2,6)$, $B(11,6)$ and $C(p,q)$.

a) Given that the point $D(6,2)$ is the midpoint of AC, determine the value of p and the value of q.

The straight line l, passes through D and is perpendicular to AC.

The point E is the intersection of l and AB.

b) Find the coordinates of E.

$\boxed{}$, $\boxed{p=10}$, $\boxed{q=-2}$, $\boxed{E(10,6)}$

(a) MIDPOINT CALCULATION

$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = (6,2)$

$\left(\frac{2+p}{2}, \frac{6+q}{2}\right) = (6,2)$

$\therefore \frac{2+p}{2} = 6 \Rightarrow p = 10$

$\frac{6+q}{2} = 2 \Rightarrow q = -2$ //

(b) GRADIENT AD

$\frac{y_2-y_1}{x_2-x_1} = \frac{6-2}{2-6} = \frac{4}{-4} = -1$

$\therefore$ GRADIENT OF l IS 1

• EQUATION OF l THROUGH $D(6,2)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 2 = 1(x-6)$

$\Rightarrow y - 2 = x - 6$

$\Rightarrow y = x - 4$

• LINE AB IS HORIZONTAL, i.e. $y=6$

THUS

$6 = x - 4$

$x = 10$

$\therefore E(10,6)$ //

Question 44 (***+)

The straight line segment joining the points with coordinates $(-7,k)$ and $(-2,-11)$, where k is a constant, has gradient -2.

a) Determine the value of k.

The midpoint of the straight line segment joining the points with coordinates $(a,-3)$ and $(4,1)$ has coordinates $(9,b)$, where a and b are constants.

b) Find the value of a and the value of b.

The straight line segment joining the points with coordinates $(-7,7)$ and $(-3,c)$, where c is a constant, has length $\sqrt{17}$.

c) Determine the possible values of c.

$\boxed{}$, $\boxed{k=-1}$, $\boxed{a=14}$, $\boxed{b=-1}$, $\boxed{c=6,\ 8}$

a) GRADIENT $= -2$

$$\Rightarrow \frac{y_2-y_1}{x_2-x_1} = -2$$

$$\Rightarrow \frac{-11-k}{-2-(-7)} = -2$$

$$\Rightarrow \frac{-11-k}{5} = -2$$

$$\Rightarrow -11-k = -10$$

$$\Rightarrow -k = 1$$

$$\Rightarrow k = -1$$

b) MIDPOINT $= (9,b)$

$$\left(\frac{a+4}{2}, \frac{-3+1}{2}\right) = (9,b)$$

$$\left\{\begin{array}{l}\frac{a+4}{2} = 9 \\ -1 = b\end{array}\right\} \Rightarrow$$

$$\Rightarrow \left\{\begin{array}{l}\therefore a = 14 \\ \therefore b = -1\end{array}\right.$$

c) DISTANCE $= \sqrt{17}$

$$\Rightarrow \sqrt{(y_2-y_1)^2+(x_2-x_1)^2} = \sqrt{17}$$

$$\Rightarrow \sqrt{(c-7)^2+(-3+7)^2} = \sqrt{17}$$

$$\Rightarrow \sqrt{(c-7)^2+16} = \sqrt{17}$$

$$\Rightarrow (c-7)^2 + 16 = 17$$

$$\Rightarrow (c-7)^2 = 1$$

$$\Rightarrow c-7 = \begin{cases} 1 \\ -1 \end{cases}$$

$$\Rightarrow c = \begin{cases} 8 \\ 6 \end{cases}$$

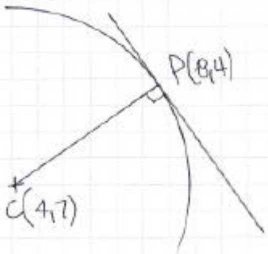
Question 26 (***+)

A circle has equation

$$x^2 + y^2 - 8x - 14y + 40 = 0$$

Find an equation of the tangent to the circle at the point $(8,4)$.

$\boxed{4x - 3y = 20}$



$x^2+y^2-8x-14y+40=0$

$x^2-8x+y^2-14y+40=0$

$(x-4)^2-16+(y-7)^2-49+40=0$

$(x-4)^2+(y-7)^2=25$ ∴ CENTRE $(4,7)$

GRADIENT $CP = \frac{4-7}{8-4} = -\frac{3}{4}$

GRADIENT OF TANGENT $= \frac{4}{3}$, $P(8,4)$

$\Rightarrow y-y_0=m(x-x_0)$

$\Rightarrow y-4=\frac{4}{3}(x-8)$

$\Rightarrow 3y-12=4x-32$

$\Rightarrow 3y=4x \;\; 20$

Question 27 (***+)

The points A, B and C have coordinates $(-3,0)$, $(-1,6)$ and $(11,2)$, respectively.

a) Show clearly that

$$\measuredangle ABC = 90°.$$

The points A, B and C lie on the circumference of a circle centred at the point D.

b) Find the coordinates of D.

c) Find an equation for this circle in the form

$$x^2 + y^2 + ax + by + c = 0,$$

where a, b and c, are constants to be found.

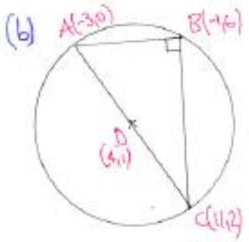
$\boxed{D(4,1)}$, $\boxed{x^2 + y^2 - 8x - 2y - 33 = 0}$

(a) GRADIENT $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{6-0}{-1+3} = \frac{6}{2} = 3$

GRADIENT $BC = \frac{y_2-y_1}{x_2-x_1} = \frac{2-6}{11+1} = -\frac{4}{12} = -\frac{1}{3}$

AS GRADIENTS ARE NEGATIVE RECIPROCALS OF EACH OTHER $A\hat{B}C = 90°$

(b)



D IS THE MIDPOINT OF AC

$D\left(\frac{-3+11}{2}, \frac{2+0}{2}\right)$

$D(4,1)$

(c) USING $d = \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

$|AD| = \sqrt{(2-1)^2+(11-4)^2}$

$|AD| = \sqrt{1+49} = \sqrt{50}$

$\therefore (x-4)^2+(y-1)^2 = (\sqrt{50})^2$

$x^2-8x+16+y^2-2y+1 = 50$

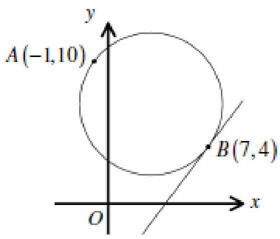
$x^2+y^2-8x-2y-33 = 0$

$\therefore a = -8$

$b = -2$

$c = -33$

Question 35 (***+)



The figure above shows a circle that passes through the points $A(-1,10)$ and $B(7,4)$.

a) Given that AB is a diameter of the circle show that an equation for this circle is given by

$$x^2+y^2-6x-14y+33=0.$$

The tangent to the circle at B meets the y axis at the point D.

b) Show that the coordinates of D are $\left(0,-\frac{16}{3}\right)$.

proof

(a) • MIDPOINT AB $= \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
$= \left(\frac{7-1}{2}, \frac{10+4}{2}\right)$
$= (3,7)$

• DISTANCE AB $= \sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$
$= \sqrt{(10-4)^2+(-1-7)^2}$
$= \sqrt{36+64}$
$= 10$

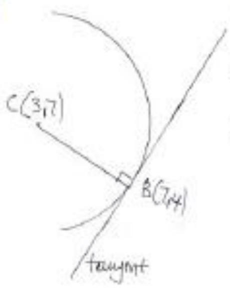
$\therefore$ CENTRE $(3,7)$ & RADIUS 5

$(y-7)^2+(x-3)^2=25$

$y^2-14y+49+x^2-6x+9=25$

$x^2+y^2-6x-14y+33=0$ // AS REQUIRED

(b)



• GRADIENT BC $= \frac{y_2-y_1}{x_2-x_1} = \frac{4-7}{7-3} = -\frac{3}{4}$

• TANGENT GRADIENT IS $\frac{4}{3}$

• $y-y_0 = m(x-x_0)$

$y-4 = \frac{4}{3}(x-7)$

$3y-12 = 4x-28$

$\boxed{3y = 4x-16}$

when $x=0$

$3y=-16$

$y=-\frac{16}{3}$

$\therefore D\left(0,-\frac{16}{3}\right)$ //

Binomial expansion – finite, infinite

Question 22 (***)

$$f(x)=(2+x)^4$$

a) Find the expansion of $f(x)$, in ascending powers of x.

b) Deduce the expansion of $(2-3x)^4$, also in ascending powers of x.

c) Determine the coefficient of x in the expansion of

$$(2+x)^4(2-3x)^4.$$

$\boxed{f(x)=16+32x+24x^2+8x^3+x^4}$, $\boxed{16-96x+216x^2-216x^3+81x^4}$, $\boxed{-1024}$

(a) $(2+x)^4 = \binom{4}{0}(2)^4(x)^0 + \binom{4}{1}(2)^3(x)^1 + \binom{4}{2}(2)^2(x)^2 + \binom{4}{3}(2)^1(x)^3 + \binom{4}{4}(2)^0(x)^4$

$= (1\times16\times1) + (4\times8\times x) + (6\times4\times x^2) + (4\times2\times x^3) + (1\times1\times x^4)$

$= 16+32x+24x^2+8x^3+x^4$

(b) $x \longmapsto (-3x)$

$\therefore f(-3x) = (2-3x)^4 = 16+32(-3x)+24(-3x)^2+8(-3x)^3+(-3x)^4$

$= 16-96x+216x^2-216x^3+81x^4$

(c) $(2+x)^4(2-3x)^4 = (16+32x+\dots)(16-96x+\dots)$

$512x$

$-1536x$

$\therefore \quad 512-1536 = -1024$

Question 33 (***+)

$$(2+x)^5 = x^5+10x^4+ax^3+bx^2+cx+32.$$

a) Find the value of each of the constants a, b and c.

b) Hence, or otherwise, simplify $\left(2-\sqrt{2}\right)^5$, giving the final answer in the form $p+q\sqrt{2}$, where p and q are constants.

$\boxed{a=40,\ b=80,\ c=80}$, $\boxed{232-164\sqrt{2}}$

(a) $(2+x)^5 = \binom{5}{0}(2)^0(x)^5 + \binom{5}{1}(2)^1(x)^4 + \binom{5}{2}(2)^2(x)^3 + \binom{5}{3}(2)^3(x)^2 + \binom{5}{4}(2)^4(x)^1 + \binom{5}{5}(2)^5(x)^0$

$= (1 \times 1 \times x^5) + (5 \times 2 \times x^4) + (10 \times 4 \times x^3) + (10 \times 8 \times x^2) + (5 \times 16 \times x) + (1 \times 32 \times 1)$

$= x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

If $a = 40$
$b = 80$
$c = 80$

(b) LET $\boxed{x = -\sqrt{2}}$

$(2-\sqrt{2})^5 = (-\sqrt{2})^5 + 10(-\sqrt{2})^4 + 40(-\sqrt{2})^3 + 80(-\sqrt{2})^2 + 80(-\sqrt{2}) + 32$

$= -4\sqrt{2} + 10(4) + 40(-2\sqrt{2}) + 80(2) - 80\sqrt{2} + 32$

$= -4\sqrt{2} + 40 - 80\sqrt{2} + 160 - 80\sqrt{2} + 32$

$= 232 - 164\sqrt{2}$

Question 40 (***+)

a) Find the binomial expansion of $\left(1+\frac{1}{4}x\right)^{10}$ in ascending powers of x up and including the term in x^3, simplifying fully each coefficient.

b) Use the expansion of part (a) to show that

$$\left(\frac{41}{40}\right)^{10} \approx 1.28 .$$

$\boxed{}$, $\boxed{1+\frac{5}{2}x+\frac{45}{16}x^2+\frac{15}{8}x^3+\dots}$

(a) $\left(1+\frac{1}{4}x\right)^{10} = 1 + \frac{10}{1}\left(\frac{1}{4}x\right) + \frac{10\times 9}{1\times 2}\left(\frac{1}{4}x\right)^2 + \frac{10\times 9\times 8}{1\times 2\times 3}\left(\frac{1}{4}x\right)^3 + \dots$

$= 1 + \frac{5}{2}x + \frac{45}{16}x^2 + \frac{15}{8}x^3 + \dots$

(b) $1+\frac{1}{4}x = \frac{41}{40}$

$\frac{1}{4}x = \frac{1}{40}$

$x = \frac{1}{10} = 0.1$

THUS FOR "SMALL" x

$\left(1+\frac{1}{4}x\right)^{10} \simeq 1 + \frac{5}{2}x + \frac{45}{16}x^2 + \frac{15}{8}x^3$

LET $x = 0.01$

$\left(\frac{41}{40}\right)^{10} \simeq 1 + \frac{5}{2}(0.1) + \frac{45}{16}(0.1)^2 + \frac{15}{8}(0.1)^3$

$\left(\frac{41}{40}\right)^{10} \simeq 1 + \frac{1}{4} + \frac{9}{320} + \frac{3}{1600}$

$\left(\frac{41}{40}\right)^{10} \simeq 1.28$

Question 5 (**+)

$$f(x) = \frac{5x+3}{(1-x)(1+3x)}, \quad |x| < \frac{1}{3}.$$

a) Express $f(x)$ into partial fractions.

b) Hence find the series expansion of $f(x)$, up and including the term in x^3.

$\boxed{}$, $\boxed{f(x) = \frac{2}{1-x} + \frac{1}{1+3x}}$, $\boxed{f(x) = 3 - x + 11x^2 - 25x^3 + O\left(x^4\right)}$

(a) $f(x) = \frac{5x+3}{(1-x)(1+3x)} = \frac{A}{1-x} + \frac{B}{1+3x}$

$\boxed{5x+3 \equiv A(1+3x) + B(1-x)}$

If $x=1 \Rightarrow 8 = 4A \Rightarrow A=2$

If $x=-\frac{1}{3} \Rightarrow \frac{4}{3} = \frac{4}{3}B \Rightarrow B=1$

$f(x) = \frac{2}{1-x} + \frac{1}{1+3x}$

(b) $f(x) = 2(1-x)^{-1} + (1+3x)^{-1}$

- $2(1-x)^{-1} = 2\left[1 + \frac{-1}{1}(-x) + \frac{(-1)(-2)}{1\times 2}(-x)^2 + \frac{(-1)(-2)(-3)}{1\times 2\times 3}(-x)^3 + O(x^4)\right]$

$= 2\left[1 + x + x^2 + x^3 + O(x^4)\right]$

$= 2 + 2x + 2x^2 + 2x^3 + O(x^4)$

REPLACED x BY $-3x$ IN PREVIOUS EXPANSION OR FIND IT FROM SCRATCH

- $(1+3x)^{-1} = 1 + (-3x) + (-3x)^2 + (-3x)^3 + O(x^4)$

$= 1 - 3x + 9x^2 - 27x^3 + O(x^4)$

ADD $f(x) = 3 - x + 11x^2 - 25x^3 + O(x^4)$

Question 9 (**+)

$$f(x)=\frac{(1+2x)^2}{1-2x}, \quad x \neq \frac{1}{2}.$$

a) Find the first 4 terms in the series expansion of $f(x)$.

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\boxed{}$, $\boxed{f(x)=1+6x+16x^2+32x^3+O(x^4)}$, $\boxed{-\frac{1}{2}<x<\frac{1}{2}}$

(a) $f(x)=\dfrac{(1+2x)^2}{1-2x}=(1+2x)^2(1-2x)^{-1}$

$$=(1+4x+4x^2)\left(1+\frac{(-1)}{1}(-2x)+\frac{(-1)(-2)}{1\times 2}(-2x)^2+\frac{(-1)(-2)(-3)}{1\times 2\times 3}(-2x)^3+\ldots\right)$$

$$=(1+4x+4x^2)(1+2x+4x^2+8x^3+\ldots)$$

$$\begin{aligned}
= 1+2x+4x^2&+8x^3+\ldots\\
4x+8x^2&+16x^3+\ldots\\
+4x^2&+8x^3+\ldots
\end{aligned}$$

$$=1+6x+16x^2+32x^3+\ldots$$

(b) VALID FOR $|2x|<1$

$|x|<\frac{1}{2}$ $\quad\therefore -\frac{1}{2}<x<\frac{1}{2}$

Question 10 (***)

$$f(x)=(1+3x)\left(1-\tfrac{2}{3}x\right)^{-2}.$$

a) Show that if x is numerically small

$$f(x)\approx 1+\frac{13}{3}x+\frac{16}{3}x^2+\frac{140}{27}x^3.$$

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\boxed{}$, $\boxed{-\frac{3}{2}<x<\frac{3}{2}}$

(a) $(1+3x)\left(1-\frac{2}{3}x\right)^{-2} = (1+3x)\left[1+\frac{-2}{1}\left(-\frac{2}{3}x\right)^1+\frac{-2(-3)}{1\times 2}\left(-\frac{2}{3}x\right)^2+\frac{-2(-3)(-4)}{1\times 2\times 3}\left(-\frac{2}{3}x\right)^3+\dots\right]$

$f(x) = (1+3x)\left[1+\frac{4}{3}x+\frac{4}{3}x^2+\frac{32}{27}x^3+\dots\right]$

$f(x) = 1+\frac{4}{3}x+\frac{4}{3}x^2+\frac{32}{27}x^3+\dots$

$\qquad\quad 3x+4x^2+4x^3+\dots$

$f(x) = 1+\frac{13}{3}x+\frac{16}{3}x^2+\frac{140}{27}x^3+\dots$

(b) VALID FOR $\left|\frac{2}{3}x\right|<1$

$|x|<\frac{3}{2}$ $\qquad -\frac{3}{2}<x<\frac{3}{2}$

Question 34 (***+)

The binomial expression $(1+12x)^{\frac{3}{4}}$ is to be expanded as an infinite convergent series, in ascending powers of x.

a) Find the first 4 terms in the expansion of $(1+12x)^{\frac{3}{4}}$.

b) State the range of values of x for which the expansion is valid.

c) By substituting a suitable value for x in the expansion show that

$$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467.$$

$\boxed{}$, $\boxed{1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3+O\left(x^4\right)}$, $\boxed{-\frac{1}{12}<x<\frac{1}{12}}$

(a) $(1+12x)^{\frac{3}{4}} = 1+\frac{3}{4}(12x)+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)}{2!}(12x)^2+\frac{\frac{3}{4}\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(12x)^3+O(x^4)$

$= 1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3+O(x^4)$

(b) VALID FOR $|12x|<1$ $\qquad -\frac{1}{12}<x<\frac{1}{12}$

(c) $\frac{53}{50} = 1+12x$

$\frac{3}{50} = 12x$

$x = 0.005$

HENCE $(1+12x)^{3/4} \approx 1+9x-\frac{27}{2}x^2+\frac{135}{2}x^3$

$(1+12(0.005))^{\frac{3}{4}} \approx 1+9(0.005)-\frac{27}{2}(0.005)^2+\frac{135}{2}(0.005)^3$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.044670938\dots$

$\left(\frac{53}{50}\right)^{\frac{3}{4}} \approx 1.04467$

Trig

Question 14 (***)

It is given that

$$\frac{1+\cot^2 x}{\cot x \operatorname{cosec} x} \equiv \sec x .$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{4\left(1+\cot^2 x\right)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5 , \quad 0 \le x < 2\pi ,$$

giving the answers in terms of π .

$\boxed{}$, $\boxed{x = \frac{\pi}{3}, \frac{5\pi}{3}}$

(a) $\text{LHS} = \frac{1+\cot^2 x}{\cot x \operatorname{cosec} x} = \frac{\operatorname{cosec}^{\cancel{2}} x}{\cot x \, \cancel{\operatorname{cosec} x}} = \frac{\operatorname{cosec} x}{\cot x} = \left(\frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}\right)$

$= \frac{\cancel{\sin x}}{\cancel{\sin x}\cos x} = \frac{1}{\cos x} = \sec x = \text{RHS}$

(b) $\frac{4(1+\cot^2 x)}{\cot x \operatorname{cosec} x} = \tan^2 x + 5$

$\Rightarrow 4\sec x = \tan^2 x + 5$

$\Rightarrow 4\sec x = (\sec^2 x - 1) + 5$

$\Rightarrow 0 = \sec^2 x - 4\sec x + 4$

$\Rightarrow 0 = (\sec x - 2)^2$

$\Rightarrow \sec x = 2$

$\Rightarrow \cos x = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\left(\begin{array}{l} x = \frac{\pi}{3} \pm 2n\pi \\ x = \frac{5\pi}{3} \pm 2n\pi \end{array}\right.$ $n = 0, 1, 2, 3, \ldots$

$x_1 = \pi/3$

$x_2 = 5\pi/3$

Question 15 (***)

It is given that

$$\frac{1-\cos 2\theta}{\sin 2\theta} \equiv \tan\theta\,, \quad \theta \neq 90k°,\, k \in \mathbb{Z}\,.$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that

$$\tan 15° = 2-\sqrt{3}\,.$$

$\boxed{}$, $\boxed{\text{proof}}$

(a) $\text{LHS} = \dfrac{1-\cos 2\theta}{\sin 2\theta} = \dfrac{1-(1-2\sin^2\theta)}{2\sin\theta\cos\theta} = \dfrac{\cancel{2}\sin^{\cancel{2}}\theta}{\cancel{2\sin\theta}\cos\theta} = \dfrac{\sin\theta}{\cos\theta}$

$= \tan\theta = \text{RHS}$

(b) LET $\boxed{\theta = 15}$

$\tan\theta \equiv \dfrac{1-\cos 2\theta}{\sin 2\theta}$

$\tan 15 = \dfrac{1-\cos 30}{\sin 30} = \dfrac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}}\,{\scriptstyle \times 2} = \dfrac{2-\sqrt{3}}{1} = 2-\sqrt{3}$

Question 17 (***)

Solve the trigonometric equation

$$\cos\theta + \sec\theta = \frac{5}{2}, \quad 0° \leq \theta < 360°\,.$$

$\boxed{\theta = 60°, 300°}$

$$\cos\theta + \sec\theta = \frac{5}{2}$$

$$\Rightarrow \cos\theta + \frac{1}{\cos\theta} = \frac{5}{2}$$

$$\Rightarrow 2\cos\theta + \frac{2}{\cos\theta} = 5$$

$$\Rightarrow 2\cos^2\theta + 2 = 5\cos\theta$$

$$\Rightarrow 2\cos^2\theta - 5\cos\theta + 2 = 0$$

$$\Rightarrow (2\cos\theta - 1)(\cos\theta - 2) = 0$$

$$\Rightarrow \cos\theta = \begin{cases} \frac{1}{2} \\ \cancel{2} \end{cases}$$

$$\Rightarrow \cos\theta = \frac{1}{2}$$

$$\arccos\left(\tfrac{1}{2}\right) = 60^\circ$$

$$\begin{cases} \theta = 60^\circ \pm 360n \\ \theta = 300 \pm 360n \end{cases} \quad n = 0,1,2,3,\ldots$$

$$\theta_1 = 60^\circ$$
$$\theta_2 = 300^\circ$$

Question 45 (***+)

It is given that

$$\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} \equiv 2\tan\theta\sec\theta.$$

a) Prove the validity of the above trigonometric identity.

b) Hence solve the equation

$$\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1}{2}\cot\theta\sec\theta, \quad 0 \le \theta < 360^\circ.$$

$\boxed{\theta = 26.6^\circ, 153.4^\circ, 206.6^\circ, 333.4^\circ}$

(a)

$$\text{LHS} = \frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1(\operatorname{cosec}\theta + 1) + 1(\operatorname{cosec}\theta - 1)}{(\operatorname{cosec}\theta - 1)(\operatorname{cosec}\theta + 1)}$$

$$= \frac{\operatorname{cosec}\theta \cancel{+1} + \operatorname{cosec}\theta \cancel{-1}}{\operatorname{cosec}^2\theta - 1} = \frac{2\operatorname{cosec}\theta}{(1+\cot^2\theta)-1} = \frac{2\operatorname{cosec}\theta}{\cot^2\theta}$$

$$= \frac{\left(\frac{2}{\sin\theta}\right)}{\left(\frac{\cos^2\theta}{\sin^2\theta}\right)} = \frac{2\sin^2\theta}{\sin\theta\cos^2\theta} = \frac{2\sin\theta}{\cos^2\theta} = \frac{2\sin\theta}{\cos\theta} \times \frac{1}{\cos\theta}$$

$$= 2\tan\theta\sec\theta = \text{RHS}$$

(b)

$$\frac{1}{\operatorname{cosec}\theta - 1} + \frac{1}{\operatorname{cosec}\theta + 1} = \frac{1}{2}\cot\theta\sec\theta$$

$$\Rightarrow 2\tan\theta\sec\theta = \frac{1}{2}\cot\theta\sec\theta$$

$$\Rightarrow 4\tan\theta\sec\theta = \cot\theta\sec\theta$$

$$\Rightarrow 4\tan\theta\sec\theta - \cot\theta\sec\theta = 0$$

$$\Rightarrow \sec\theta(4\tan\theta - \cot\theta) = 0$$

$$\Rightarrow \frac{1}{\cos\theta}(4\tan\theta - \cot\theta) = 0$$

$$\Rightarrow 4\tan\theta - \cot\theta = 0$$

$$\Rightarrow 4\tan\theta = \cot\theta$$

$$\Rightarrow 4\tan\theta = \frac{1}{\tan\theta}$$

$$\Rightarrow 4\tan^2\theta = 1$$

$$\Rightarrow \tan^2\theta = \frac{1}{4}$$

$$\Rightarrow \tan\theta = \begin{cases} \tfrac{1}{2} \\ -\tfrac{1}{2} \end{cases}$$

- $\arctan\left(\frac{1}{2}\right) = 26.6°$, $\theta = 26.6 \pm 360n$
- $\arctan\left(-\frac{1}{2}\right) = -26.6°$, $\theta = -26.6 \pm 360n$

$n = 0, 1, 2, 3, \ldots$

Hence $\theta_1 = 26.6°$, $\theta_2 = 206.6°$, $\theta_3 = 153.4°$, $\theta_4 = 333.4°$

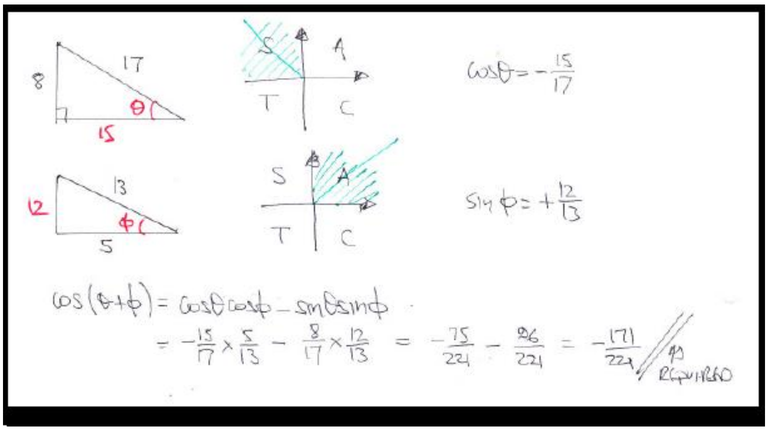
Question 32 (***+)

$$\sin\theta = \frac{8}{17} \quad \text{and} \quad \cos\varphi = \frac{5}{13}.$$

If θ is obtuse and φ is acute, show that

$$\cos(\theta + \varphi) = -\frac{171}{221}.$$

$\boxed{}$, $\boxed{\text{proof}}$



Question 33 (***+)

A curve C is defined by the equation

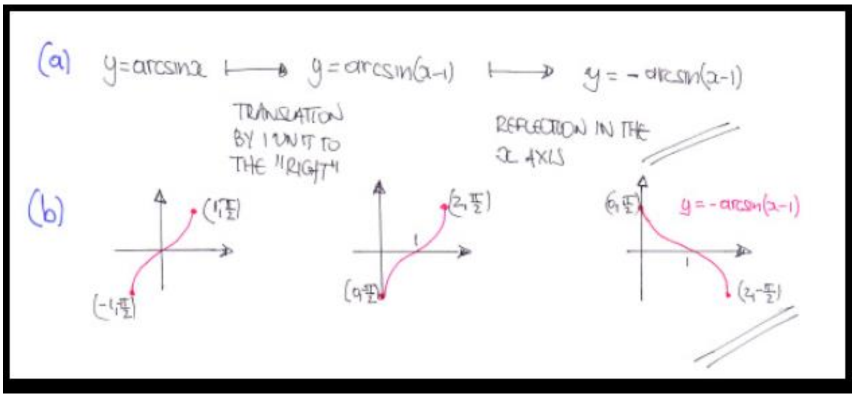
$$y = -\arcsin(x-1), \ 0 \le x \le 2.$$

a) Describe the 2 geometric transformation that map the graph of $\arcsin x$ onto the graph of C.

b) Sketch the graph of C.

The sketch must include the coordinates of any points where the graph of C meets the coordinate axes and the coordinates of the endpoints of C.

$\boxed{}$, $\boxed{\text{translation by 1 unit to the right, followed by reflection in the } x \text{ axis}}$



Question 38 (***+)

$$y \equiv \sqrt{2}\cos\theta - \sqrt{6}\sin\theta,\ 0 < \theta < 360°.$$

a) Express y in the form $R\cos(\theta + \alpha)$, $R > 0$, $0 < \alpha < 90°$.

b) Solve the equation $y = 2$.

c) Write down the minimum value of …

i. … y^2.

ii. … $\dfrac{1}{y^2}$.

$\boxed{}$, $\boxed{y \equiv \sqrt{8}\cos(\theta + 60°)}$, $\boxed{\theta = 255°, 345°}$, $\boxed{\min = 0}$, $\boxed{\min = \frac{1}{8}}$

(a) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta \equiv R\cos(\theta+\alpha)$

$\equiv R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$

$\equiv (R\cos\alpha)\cos\theta - (R\sin\alpha)\sin\theta$

$\left.\begin{array}{l} R\cos\alpha = \sqrt{2} \\ R\sin\alpha = \sqrt{6} \end{array}\right\} \begin{array}{l} \Rightarrow R = \sqrt{6+2} = \sqrt{8} \\ \Rightarrow \tan\alpha = \frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3} \quad \therefore \alpha = 60° \end{array}$

$\therefore y = \sqrt{8}\cos(\theta + 60°)$

(b) $\sqrt{2}\cos\theta - \sqrt{6}\sin\theta = 2$

$\Rightarrow \sqrt{8}\cos(\theta+60°) = 2$

$\Rightarrow \cos(\theta+60°) = \frac{\sqrt{2}}{2}$

$\Rightarrow \arccos\left(\frac{\sqrt{2}}{2}\right) = 45$

$\theta + 60 = 45 \pm 360n$

$\theta + 60 = 315 \pm 360n \qquad n = 0, 1, 2, 3, \ldots$

$\theta = -15 \pm 360n$

$\theta = 255 \pm 360n$

$\therefore \theta_1 = 345°$, $\theta_2 = 255°$

(c) $y^2 = \left[\sqrt{8}\cos(\theta+60)\right]^2 = 8\cos^2(\theta + 60°)$

$\therefore y^2_{MIN} = 0$

(d) THE MINIMUM VALUE OF $\frac{1}{y^2}$ OCCURS WHEN y^2 IS MAXIMUM IE 8

$\therefore \left(\frac{1}{y^2}\right)_{MIN} = \frac{1}{8}$

Question 43 (***+)

It is given that

$$\cos(x + 30°) + \cos(x - 30°) \equiv \sqrt{3}\cos x .$$

a) Prove the validity of the above trigonometric identity.

b) Hence show that

$$\cos 75° + \cos 15° = \tfrac{1}{2}\sqrt{6} .$$

$\boxed{\text{proof}}$

(a) LHS $= \cos(x+30°) + \cos(x-30°)$

$= \cos x\cos 30° - \cancel{\sin x\sin 30°} + \cos x\cos 30° + \cancel{\sin x\sin 30°}$

$= 2\cos x\cos 30° = \cancel{2}\cos x \times \frac{\sqrt{3}}{\cancel{2}} = \sqrt{3}\cos x$ = RHS

(b) LET $x = 45°$ IN $\cos(x+30°) + \cos(x-30°) = \sqrt{3}\cos x$

$\cos(45°+30°) + \cos(45°-30°) = \sqrt{3}\cos 45°$

$\cos 75° + \cos 15° = \sqrt{3} \times \frac{\sqrt{2}}{2}$

$\cos 75° + \cos 15° = \frac{1}{2}\sqrt{6}$

AS REQUIRED

Question 44 (***+)

Prove the validity of each of the following trigonometric identities.

a) $\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2+\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2\equiv 4\tan^2\theta+2$.

b) $\sin^2\left(\theta+\dfrac{\pi}{4}\right)+\sin^2\left(\theta-\dfrac{\pi}{4}\right)\equiv 1$.

$\boxed{\text{proof}}$

(a) $\text{LHS}=\left(\dfrac{1+\sin\theta}{\cos\theta}\right)^2+\left(\dfrac{1-\sin\theta}{\cos\theta}\right)^2=\dfrac{1+\cancel{2\sin\theta}+\sin^2\theta}{\cos^2\theta}+\dfrac{1\cancel{-2\sin\theta}+\sin^2\theta}{\cos^2\theta}$

$=\dfrac{2+2\sin^2\theta}{\cos^2\theta}=\dfrac{2}{\cos^2\theta}+\dfrac{2\sin^2\theta}{\cos^2\theta}=2\sec^2\theta+2\tan^2\theta$

$=2(\tan^2\theta+1)+2\tan^2\theta=4\tan^2\theta+2=\text{RHS}$

(b) $\text{LHS}=\sin^2\left(\theta+\frac{\pi}{4}\right)+\sin^2\left(\theta-\frac{\pi}{4}\right)=\left[\sin\left(\theta+\frac{\pi}{4}\right)\right]^2+\left[\sin\left(\theta-\frac{\pi}{4}\right)\right]^2$

$=\left(\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}\right)^2+\left(\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}\right)^2$

$=\left(\frac{\sqrt{2}}{2}\sin\theta+\frac{\sqrt{2}}{2}\cos\theta\right)^2+\left(\frac{\sqrt{2}}{2}\sin\theta-\frac{\sqrt{2}}{2}\cos\theta\right)^2$

$=\frac{1}{2}(\sin\theta+\cos\theta)^2+\frac{1}{2}(\sin\theta-\cos\theta)^2=\frac{1}{2}\left[(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2\right]$

$=\frac{1}{2}\left[\sin^2\theta+\cancel{2\sin\theta\cos\theta}+\cos^2\theta+\sin^2\theta-\cancel{2\sin\theta\cos\theta}+\cos^2\theta\right]$

$=\frac{1}{2}\left[2\sin^2\theta+2\cos^2\theta\right]=\sin^2\theta+\cos^2\theta=1=\text{RHS}$

ALTERNATIVE USE IDENTITY $\sin A=\cos\left(\frac{\pi}{2}-A\right)$

23 Find, to the nearest tenth of a degree, all solutions in the interval $[-360°, 360°]$ of the equation

$$3\cos x + 4\sin x = \frac{5\sqrt{3}}{2}$$

24 Express $2\cos 4x + 3\sin 4x$ in the form $R\cos(4x - \alpha)$, where $R > 0$ and α lies in the interval $\left[0, \frac{\pi}{2}\right]$.

Hence find:

(a) the minimum value of

$$\frac{1}{2\cos 4x + 3\sin 4x}$$

and the smallest positive value of x when this occurs.

(b) the values of x, in radians to 2 decimal places, for which $2\cos 4x + 3\sin 4x = 2$ in the interval $[0, 2\pi]$.

25 Express $4\sin x + 3\cos x$ in the form $R\sin(x + \alpha)$ where $R > 0$ and α lies in the interval $\left[0, \frac{\pi}{2}\right]$.

(a) Find the greatest and the least values of

$$\frac{1}{4\sin x + 3\cos x + 12}$$

(b) Determine, in radians to 2 decimal places, the values of x in the interval $[0, 2\pi]$ for which

$$(4\sin x + 3\cos x)^2 = \tfrac{1}{2}$$

23 $-336.9°, -276.9°, 23.1°, 83.1°$

24 (a) $-\dfrac{1}{\sqrt{13}}, 1.031$

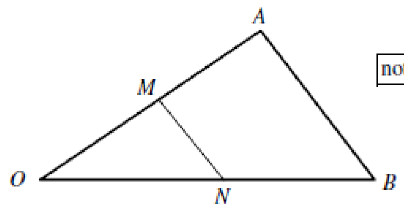
(b) 0, 0.49, 2.06, 3.63, 5.20, 1.57, 3.14, 4.71, 6.28

25 (a) $\frac{1}{7}, \frac{1}{17}$ (b) 2.36, 2.64, 5.50, 5.78

Vectors

Question 5 (**)

The figure below shows the triangle OAB.



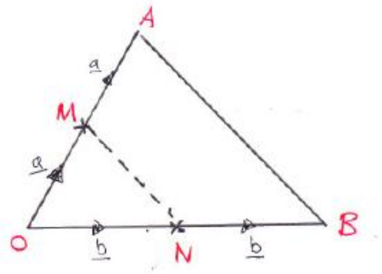
not drawn accurately

The point M is the midpoint of OA and the point N is the midpoint of OB.

Let $\overline{OM} = \mathbf{a}$ and $\overline{ON} = \mathbf{b}$.

By finding simplified expressions for $\overline{MN}$ and $\overline{AB}$, in terms of $\mathbf{a}$ and $\mathbf{b}$, show that MN is parallel to AB, and half its length.

proof



LET $\overrightarrow{OM} = \underline{a}$
$\overrightarrow{ON} = \underline{b}$

- $\overrightarrow{MN} = \overrightarrow{MO} + \overrightarrow{ON} = -\underline{a} + \underline{b} = \underline{b} - \underline{a}$
- $\overrightarrow{AB} = \overrightarrow{AM} + \overrightarrow{MO} + \overrightarrow{ON} + \overrightarrow{NB} = -\underline{a} - \underline{a} + \underline{b} + \underline{b} = 2\underline{b} - 2\underline{a} = 2(\underline{b} - \underline{a})$

  $\left(\text{OR } \overrightarrow{AM} + \overrightarrow{MN} + \overrightarrow{NB} = -\underline{a} + (\underline{b} - \underline{a}) + \underline{b} = 2\underline{b} - 2\underline{a} = 2(\underline{b} - \underline{a})\right)$

THUS $\overrightarrow{MN}$ IS IN THE DIRECTION AS $\overrightarrow{AB}$ $\therefore$ MN $\parallel$ AB

AND $\overrightarrow{AB} = 2(\underline{b} - \underline{a}) = 2\overrightarrow{MN}$ $\therefore$ MN IS HALF OF AB

Question 22 (***)

$OABC$ is a parallelogram and the point M is the midpoint of AB.

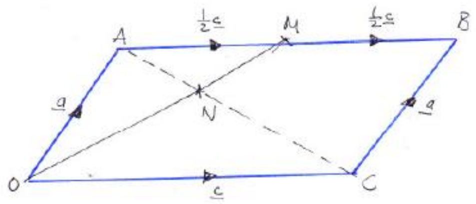
The point N lies on the diagonal AC so that $AN : NC = 1:2$.

Let $\overline{OA} = \mathbf{a}$ and $\overline{OC} = \mathbf{c}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{c}$, for each of the vectors $\overline{AC}$, $\overline{AN}$, $\overline{ON}$ and $\overline{NM}$.

b) Deduce, showing your reasoning, that O, N and M are collinear.

$\boxed{\overline{AC} = \mathbf{c} - \mathbf{a}}$, $\boxed{\overline{AN} = \frac{1}{3}\mathbf{c} - \frac{1}{3}\mathbf{a}}$, $\boxed{\overline{ON} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{c}}$, $\boxed{\overline{NM} = \frac{1}{3}\mathbf{a} + \frac{1}{6}\mathbf{c}}$



$\overrightarrow{OA} = \underline{a}$
$\overrightarrow{OC} = \underline{c}$

(a)
- $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC} = -\underline{a} + \underline{c} = \underline{c} - \underline{a}$
- $\overrightarrow{AN} = \frac{1}{3}\overrightarrow{AC} = \frac{1}{3}(\underline{c} - \underline{a}) = \frac{1}{3}\underline{c} - \frac{1}{3}\underline{a}$
- $\overrightarrow{ON} = \overrightarrow{OA} + \overrightarrow{AN} = \underline{a} + \frac{1}{3}\underline{c} - \frac{1}{3}\underline{a} = \frac{2}{3}\underline{a} + \frac{1}{3}\underline{c}$
- $\overrightarrow{NM} = \overrightarrow{NA} + \overrightarrow{AM} = -\overrightarrow{AN} + \overrightarrow{AM} = -\frac{1}{3}\underline{c} + \frac{1}{3}\underline{a} + \frac{1}{2}\underline{c} = \frac{1}{3}\underline{a} + \frac{1}{6}\underline{c}$

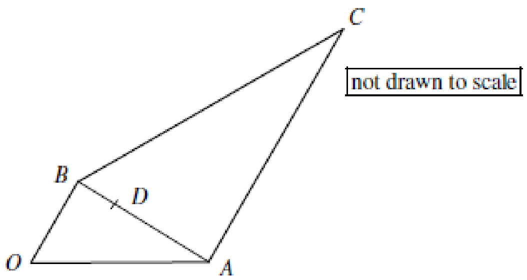
(b)
$$\left.\begin{aligned} \overrightarrow{ON} &= \tfrac{2}{3}\underline{a} + \tfrac{1}{3}\underline{c} = \tfrac{1}{3}(2\underline{a} + \underline{c}) \\ \overrightarrow{NM} &= \tfrac{1}{3}\underline{a} + \tfrac{1}{6}\underline{c} = \tfrac{1}{6}(2\underline{a} + c) \end{aligned}\right\} \Rightarrow$$

$\overrightarrow{ON}$ IS "PARALLEL" TO $\overrightarrow{NM}$
AND THEY SHARE A POINT

$\therefore$ THE LIE ON A STRAIGHT LINE

Question 30 (***)

The figure below shows a trapezium $OBCA$ where OB is parallel to AC.



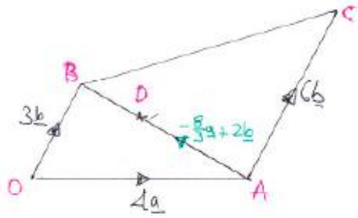
The point D lies on BA so that $BD:DA = 1:2$.

Let $\overline{OA} = 4\mathbf{a}$, $\overline{OB} = 3\mathbf{b}$ and $\overline{AC} = 6\mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overline{OC}$, $\overline{AB}$, $\overline{AD}$ and $\overline{OD}$.

b) Deduce, showing your reasoning, that O, D and C are collinear and state the ratio of $OD:DC$.

$\boxed{\overline{OC} = 4\mathbf{a} + 6\mathbf{b}}$, $\boxed{\overline{AB} = -4\mathbf{a} + 3\mathbf{b}}$, $\boxed{\overline{AD} = -\frac{8}{3}\mathbf{a} + 2\mathbf{b}}$, $\boxed{\overline{OD} = \frac{4}{3}\mathbf{a} + 2\mathbf{b}}$, $\boxed{OD:DC = 1:2}$



(a)

- $\vec{OC} = \vec{OA} + \vec{AC} = 4a + 6b$
- $\vec{AB} = \vec{AO} + \vec{OB} = -4a + 3b$
- $\vec{AD} = \frac{2}{3}\vec{AB} = \frac{2}{3}(-4a + 3b)$
  $= -\frac{8}{3}a + 2b$
- $\vec{OD} = \vec{OA} + \vec{AD}$
  $\vec{OD} = 4a + \left(-\frac{8}{3}a + 2b\right)$
  $\vec{OD} = \frac{4}{3}a + 2b$

(b) $\vec{DC} = \vec{DA} + \vec{AC} = \left(\frac{8}{3}a - 2b\right) + 6b = \frac{8}{3}a + 4b = 2\left(\frac{4}{3}a + 2b\right)$

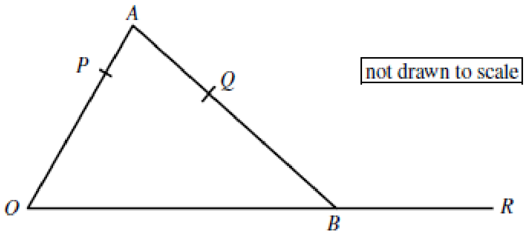
$\vec{OD} = \frac{4}{3}a + 2b$

AS VECTORS $\vec{DC}$ & $\vec{OD}$ ARE IN THE SAME DIRECTION AND SHARE D, THE POINTS O, D, C ARE COLLINEAR

RATIO OD : DC = 1 : 2

Question 33 (***)

The figure below shows a triangle OAB.



- The point P lies on OA so that $OP : PA = 4:1$.
- The point Q lies on AB so that $AQ : QB = 2:3$
- The side OB is extended to the point R so that $OB : BR = 5:3$.

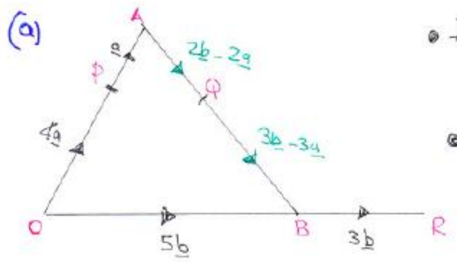
Let $\overrightarrow{PA} = \mathbf{a}$ and $\overrightarrow{OB} = 5\mathbf{b}$.

a) Find simplified expressions, in terms of $\mathbf{a}$ and $\mathbf{b}$, for each of the vectors $\overrightarrow{AB}$, $\overrightarrow{AQ}$ and $\overrightarrow{PQ}$.

b) Deduce, showing your reasoning, that P, Q and R are collinear and state the ratio of $PQ : QR$.

$\boxed{\overrightarrow{AB} = 5\mathbf{b} - 5\mathbf{a}}$, $\boxed{\overrightarrow{AQ} = 2\mathbf{b} - 2\mathbf{a}}$, $\boxed{\overrightarrow{PQ} = 2\mathbf{b} - \mathbf{a}}$, $\boxed{PQ : QR = 1:3}$

(a)



- $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB} = -a - 4a + 5b$
  $= -5a + 5b = 5b - 5a$
- $\overrightarrow{AQ} = \frac{2}{5}\overrightarrow{AB} = \frac{2}{5}(5b - 5a)$
  $= 2b - 2a$

thus

- $\overrightarrow{PQ} = \overrightarrow{PA} + \overrightarrow{AQ}$
  $= a + (2b - 2a)$
  $= -a + 2b$
  $= 2b - a$

(b) $\overrightarrow{PQ} = 2b - a$ (found)

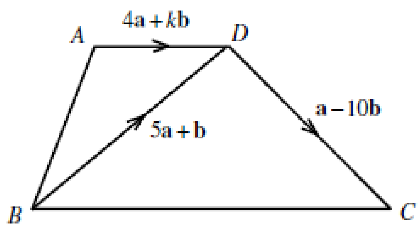
$\overrightarrow{QR} = \overrightarrow{QB} + \overrightarrow{BR} = 3b - 3a + 3b = 6b - 3a = 3(2b - a)$

VECTORS $\overrightarrow{PQ}$ & $\overrightarrow{QR}$ ARE IN THE SAME DIRECTION & SHARE POINT Q

∴ POINTS P, Q, R ARE COLLINEAR

& RATIO $PQ : QR = 1:3$

Question 44 (***+)



The figure above shows a trapezium $ABCD$ where AD is parallel to BC.

The following information is given for this trapezium.

$$\overrightarrow{BD} = 5\mathbf{a} + \mathbf{b}, \qquad \overrightarrow{DC} = \mathbf{a} - 10\mathbf{b} \qquad \text{and} \qquad \overrightarrow{AD} = 4\mathbf{a} + k\mathbf{b},$$

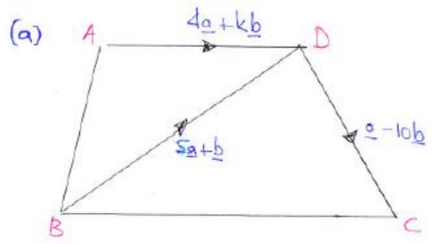
where k is an integer.

a) Find the value of k.

b) Find a simplified expression for $\overrightarrow{AB}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

$\boxed{k = -6}$, $\boxed{\overrightarrow{AB} = -\mathbf{a} - 7\mathbf{b}}$

(a)



$\overrightarrow{BC} = \overrightarrow{BD} + \overrightarrow{DC}$
$= 5\underline{a} + \underline{b} + \underline{a} - 10\underline{b}$
$= 6\underline{a} - 9\underline{b}$

$\left.\begin{array}{l}\overrightarrow{AD} = 4\underline{a} + k\underline{b} \\ \overrightarrow{BC} = 6\underline{a} - 9\underline{b}\end{array}\right\}$ THESE ARE PARALLEL

$\therefore \frac{4}{6} = \frac{k}{-9}$

$\Rightarrow 6k = -36$

$\therefore k = -6$

(b) $\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB} = 4\underline{a} - 6\underline{b} - 5\underline{a} - \underline{b} = -\underline{a} - 7\underline{b}$

Question 12 (***)

Relative to a fixed origin O, the horizontal unit vectors $\mathbf{i}$ and $\mathbf{j}$ are pointing due east and due north, respectively.

A particle P is moving with constant acceleration of $(-\mathbf{i}+\mathbf{j})$ ms^{-2}.

It is initially observed passing through the point with position vector $-2\mathbf{j}$ m with velocity of $2\mathbf{i}$ ms^{-1}.

a) Find the speed of P, 8 s after it was first observed.

b) Calculate the distance of P from the origin, 8 s after it was first observed.

$\boxed{|\mathbf{v}| = 10 \text{ ms}^{-1}}$, $\boxed{d = 34 \text{ m}}$

(a) USING $\underline{v} = \underline{u} + \underline{a}t$

$\underline{v} = -2\underline{j} + (-\underline{i}+\underline{j}) \times 8$

$\underline{v} = -2\underline{j} - 8\underline{i} + 8\underline{j}$

$\underline{v} = -8\underline{i} + 6\underline{j}$

SPEED $= |\underline{v}| = |-8\underline{i} + 6\underline{j}|$

$= \sqrt{(-8)^2 + 6^2}$

$= \sqrt{100}$

$= 10 \text{ ms}^{-1}$

(b) USING $\underline{r} = \underline{r}_0 + \underline{u}t + \frac{1}{2}\underline{a}t^2$

$\underline{r} = -2\underline{j} + 2\underline{i} \times 8 + \frac{1}{2}(-\underline{i}+\underline{j}) \times 8^2$

$\underline{r} = -2\underline{j} + 16\underline{i} - 32\underline{i} + 32\underline{j}$

$\underline{r} = -16\underline{i} + 30\underline{j}$

$d = |\underline{r}| = |-16\underline{i} + 30\underline{j}| = \sqrt{(-16)^2 + 30^2}$

$= \sqrt{256 + 900} = \sqrt{1156}$

$= 34 \text{ m}$

Question 15 (****)

Relative to a fixed origin O, the horizontal unit vectors $\mathbf{i}$ and $\mathbf{j}$ are pointing due east and due north, respectively.

The velocity of a particle, $\mathbf{v}$ ms^{-1}, at time t s after a given instant is

$$\mathbf{v} = (3-2t)\mathbf{i} + (3t-6)\mathbf{j}.$$

a) Find the speed of the particle when $t = 0$.

b) Determine the bearing on which the particle is moving when $t = 4$.

c) Calculate the value of t when the particle is moving …

   i. … parallel to $\mathbf{i}$.

   ii. … parallel to $5\mathbf{i} - 7\mathbf{j}$.

$\boxed{\text{speed} = \sqrt{45} \approx 6.71 \text{ ms}^{-1}}$, $\boxed{320°}$, $\boxed{t=2}$, $\boxed{t=9}$

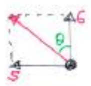
a) $\underline{v} = (3-2t)\underline{i} + (3t-6)\underline{j}$

$t=0 \quad \underline{v} = 3\underline{i} - 6\underline{j}$

$|\underline{v}| = \sqrt{3^2+(-6)^2} = \sqrt{9+36} = \sqrt{45} \approx 6.71 \text{ ms}^{-1}$

b) $t=4 \quad \underline{v} = (3-2\times4)\underline{i} + (3\times4-6)\underline{j}$

$\underline{v} = -5\underline{i} + 6\underline{j}$

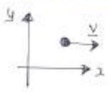


$\tan\theta = \frac{5}{6}$

$\theta \approx 39.8°$

$\therefore$ BEARING OF $360 - 39.8 \approx 320°$

c) (I) "PARALLEL" TO $\underline{i}$



$\therefore$ $\underline{j}$ COMPONENT MUST BE ZERO

$3t-6=0$

$t=2$

"PARALLEL" TO $5\underline{i} - 7\underline{j}$

$\frac{3-2t}{3t-6} = \frac{5}{-7}$

$\Rightarrow -21 + 14t = 15t - 30$

$\Rightarrow 9 = t$

IE $t=9$

Differentiation

Question 14 (**)

The curve C has equation

$$y = \frac{6}{x^2} + \frac{5x}{4} - 4, \; x \neq 0.$$

a) Find an expression for $\frac{dy}{dx}$.

b) Determine an equation of the normal to the curve at the point where $x = 2$.

$\boxed{\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}}$, $\boxed{y = 4x - 8}$

(a) $y = \frac{6}{x^2} + \frac{5x}{4} - 4$

$y = 6x^{-2} + \frac{5}{4}x - 4$

$\frac{dy}{dx} = -12x^{-3} + \frac{5}{4}$

(OR) $\frac{dy}{dx} = \frac{5}{4} - \frac{12}{x^3}$

(b) • WHEN $x = 2$

$y = \frac{6}{2^2} + \frac{5 \times 2}{4} - 4$

$y = \frac{3}{2} + \frac{5}{2} - 4$

$y = 0$

$\therefore (2,0)$

• $\left.\frac{dy}{dx}\right|_{x=2} = \frac{5}{4} - \frac{12}{2^3} = \frac{5}{4} - \frac{12}{8}$

$= \frac{5}{4} - \frac{6}{4} = -\frac{1}{4}$

$\therefore$ NORMAL GRADIENT IS 4, $(2,0)$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 0 = 4(x - 2)$

$\Rightarrow y = 4x - 8$

Question 25 (**+)

$$y = \frac{1}{x} - \frac{1}{\sqrt{x}}, \; x \in \mathbb{R}, \; x > 0.$$

Show that the value of $\frac{d^2y}{dx^2}$ where $x = 4$, is $\frac{1}{128}$.

$\boxed{\text{proof}}$

$$y = \frac{1}{x} - \frac{1}{\sqrt{x}} = x^{-1} - x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -x^{-2} + \frac{1}{2}x^{-\frac{3}{2}}$$

$$\frac{d^2y}{dx^2} = 2x^{-3} - \frac{3}{4}x^{-\frac{5}{2}} = \frac{2}{x^3} - \frac{3}{4x^{\frac{5}{2}}}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = \frac{2}{4^3} - \frac{3}{4\times 4^{\frac{5}{2}}} = \frac{2}{64} - \frac{3}{4\times 32}$$

$$= \frac{1}{32} - \frac{3}{128} = \frac{4}{128} - \frac{3}{128} = \frac{1}{128}$$

AS REQUIRED

Question 40 (***)

The curve C has equation

$$f(x) = 4x\sqrt{x} - \frac{25x^2}{16}, \quad x \geq 0.$$

a) Find a simplified expression for $f'(x)$.

b) Determine an equation of the tangent to C at the point where $x = 4$, giving the answer in the form $ax + by = c$, where a, b and c are integers.

$\boxed{}$, $\boxed{f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x}$, $\boxed{x + 2y = 18}$

(a) $f(x) = 4x\sqrt{x} - \frac{25x^2}{16}$

$\Rightarrow f(x) = 4x^1x^{\frac{1}{2}} - \frac{25}{16}x^2$

$\Rightarrow f(x) = 4x^{\frac{3}{2}} - \frac{25}{16}x^2$

$\Rightarrow f'(x) = 6x^{\frac{1}{2}} - \frac{25}{8}x$

(b) $f'(4) = 6\times 4^{\frac{1}{2}} - \frac{25}{8}\times 4$

$= 6\times 2 - \frac{25}{2}$

$= 12 - \frac{25}{2}$

$= \frac{24}{2} - \frac{25}{2}$

$= -\frac{1}{2}$

when $x = 4$

$y = f(4) = 4\times 4\times\sqrt{4} - \frac{25}{16}\times 4^2$

$= 32 - 25$

$= 7$

$\therefore (4,7)$ & GRADIENT $-\frac{1}{2}$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - 7 = -\frac{1}{2}(x - 4)$

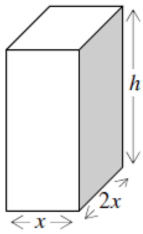
$\Rightarrow 2y - 14 = -x + 4$

$\Rightarrow 2y + x = 18$

Question 45 (***)

The figure below shows the design of a fruit juice carton with capacity of 1000 cm^3.

The design of the carton is that of a closed cuboid whose base measures x cm by $2x$ cm, and its height is h cm.



a) Show that the surface area of the carton, $A \text{ cm}^2$, is given by

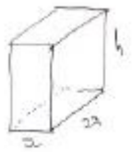
$$A = 4x^2 + \frac{3000}{x}.$$

b) Find the value of x for which A is stationary.

c) Calculate the minimum value for A, justifying fully the fact that it is indeed the minimum value of A.

$\boxed{x = \sqrt[3]{375} \approx 7.21}$, $\boxed{A_{\min} \approx 624}$

(a)



Volume/capacity $= 1000$

$2x^2h = 1000$

$x^2h = 500$

$\boxed{h = \frac{500}{x^2}}$

$\therefore A = (2x^2 + xh + 2xh) \times 2$

$\Rightarrow A = (2x^2 + 3xh) \times 2$

$\Rightarrow A = 4x^2 + 6xh$

$\Rightarrow A = 4x^2 + 6x\left(\frac{500}{x^2}\right)$

$\Rightarrow A = 4x^2 + \frac{3000}{x}$ // As required

(b) $A = 4x^2 + 3000x^{-1}$

$\frac{dA}{dx} = 8x - 3000x^{-2}$

SOLVING FOR ZERO

$\Rightarrow 8x - 3000x^{-2} = 0$

$\Rightarrow 8x - \frac{3000}{x^2} = 0$

$\Rightarrow 8x = \frac{3000}{x^2}$

$\Rightarrow 8x^3 = 3000$

$\Rightarrow x^3 = 375$

$\Rightarrow x = \sqrt[3]{375} \approx 7.21$ //

(c) when $x \approx 7.21$

$A \approx 4(7.21)^2 + \frac{3000}{7.21}$

$A \approx 624$ //

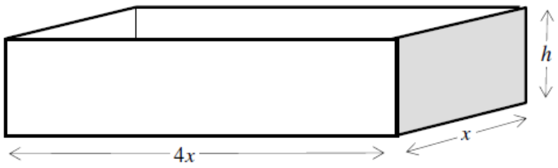
- $\frac{dA}{dx} = 8x - 3000x^{-2}$

$\frac{d^2A}{dx^2} = 8 + 6000x^{-3} = 8 + \frac{6000}{x^3}$

$\left.\frac{d^2A}{dx^2}\right|_{x=7.21} = 24 > 0$

$\therefore A$ MINIMUM //

Question 59 (***)



The figure above shows a box in the shape of a cuboid with a rectangular base x cm by $4x$ cm and **no top**. The height of the box is h cm.

It is given that the surface area of the box is 1728 cm^2.

a) Show clearly that

$$h = \frac{864 - 2x^2}{5x}.$$

b) Use part (a) to show that the volume of the box, $V \text{ cm}^3$, is given by

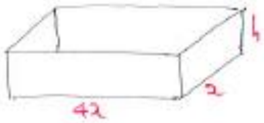
$$V = \frac{8}{5}\left(432x - x^3\right).$$

c) Find the value of x for which V is stationary.

d) Find the maximum value for V, justifying the fact that it is the maximum.

$\boxed{x = 12}$, $\boxed{V_{max} = 5529.6}$

(a)



$1728 = 4x^2 + 2(xh) + 2(4xh)$

$1728 = 4x^2 + 2xh + 8xh$

$1728 = 4x^2 + 10xh$

$864 = 2x^2 + 5xh$

$864 - 2x^2 = 5xh$

$h = \dfrac{864 - 2x^2}{5x}$ // AS REQUIRED

(b) $V = 4x^2h$

$V = 4x^2 \times \dfrac{864 - 2x^2}{5x}$

$V = \dfrac{4x(864 - 2x^2)}{5}$

$V = \dfrac{8x(432 - x^2)}{5}$

$V = \dfrac{8}{5}(432x - x^3)$ // AS REQUIRED

(c) $\dfrac{dV}{dx} = \dfrac{8}{5}(432 - 3x^2)$

FOR MIN/MAX $\dfrac{dV}{dx} = 0$

$\dfrac{8}{5}(432 - 3x^2) = 0$

$432 - 3x^2 = 0$

$3x^2 = 432$

$x^2 = 144$

$x = 12$ // $(x > 0)$

(d) $\dfrac{d^2V}{dx^2} = \dfrac{8}{5}(-6x) = -\dfrac{48}{5}x$

$\left.\dfrac{d^2V}{dx^2}\right|_{x=12} = -\dfrac{48 \times 12}{5} = -\dfrac{576}{5} < 0$

INDEED A MAXIMUM

USING $x = 12$

$V_{max} = \dfrac{8}{5}(432 \times 12 - 12^3)$

$V_{max} = \dfrac{8}{5} \times 3456$

$V_{max} = 5529.6 \text{ cm}^3$ //

Question 81 (***)

The total cost C, in £, for a certain car journey, is modelled by

$$C = \frac{200}{V} + \frac{2V}{25}, \quad V > 30,$$

where V is the average speed in miles per hour.

a) Find the value of V for which C is stationary.

b) Justify that this value of V minimizes C.

c) Hence determine the minimum total cost of the journey.

$\boxed{}$, $\boxed{V = 50}$, $\boxed{£8}$

(a) $C = \frac{200}{V} + \frac{2V}{25} = 200V^{-1} + \frac{2}{25}V$

$\frac{dC}{dV} = -200V^{-2} + \frac{2}{25}$

SOLVE FOR ZERO

$\Rightarrow -200V^{-2} + \frac{2}{25} = 0$

$\Rightarrow -\frac{200}{V^2} + \frac{2}{25} = 0$

$\Rightarrow \frac{2}{25} = \frac{200}{V^2}$

$\Rightarrow 2V^2 = 5000$

$\Rightarrow V^2 = 2500$

$\Rightarrow V = 50 \quad (V > 30)$

(b) $\frac{d^2C}{dV^2} = 400V^{-3}$

$\frac{d^2C}{dV^2} = \frac{400}{V^3}$

$\left.\frac{d^2C}{dV^2}\right|_{V=50} = \frac{400}{50^3} = \frac{2}{625} > 0$

INDEED MINIMUM

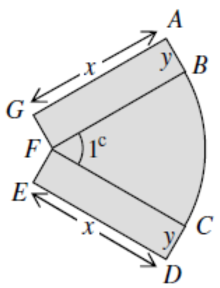
(c) WHEN $V=50$

$C = \frac{200}{50} + \frac{2\times 50}{25}$

$C = 4 + 4$

$C = £8$

Question 82 (***)



The figure above shows a clothes design consisting of two identical rectangles attached to each of the straight sides of a circular sector of radius x cm.

The rectangles measure x cm by y cm and the circular sector subtends an angle of one radian at the centre.

The perimeter of the design is 40 cm.

a) Show that the area of the design, A cm^2, is given by

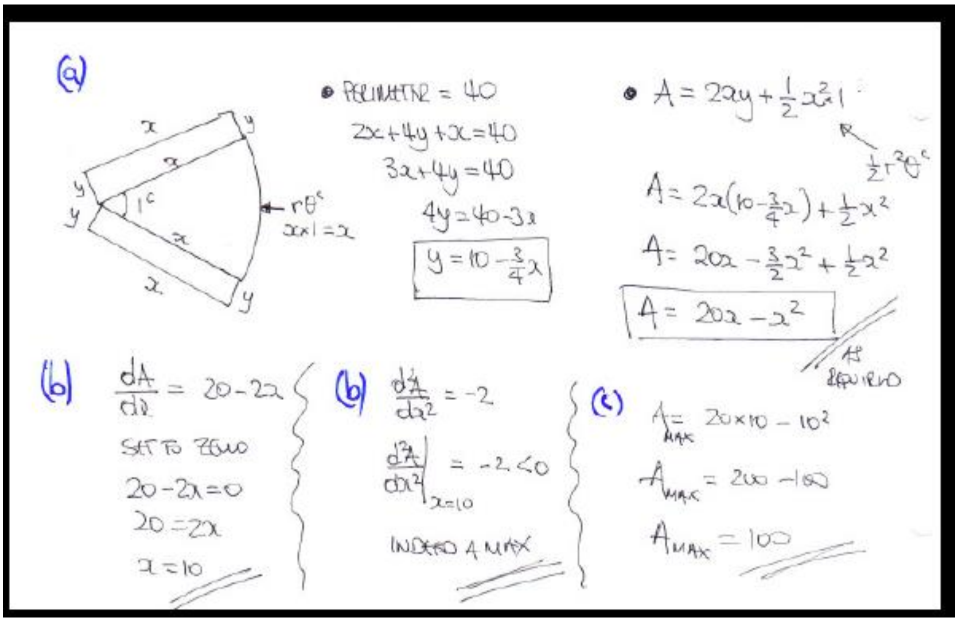
$$A = 20x - x^2 .$$

b) Determine <u>by differentiation</u> the value of x for which A is stationary.

c) Show that the value of x found in part (b) gives the maximum value for A .

d) Find the maximum area of the design.

$\boxed{x = 10}$, $\boxed{A_{max} = 100}$



Question 87 (***+)

The curve C has equation

$$y=2x^3-9x^2+12x-10.$$

a) Find the coordinates of the two points on the curve where the gradient is zero.

The point P lies on C and its x coordinate is -1.

b) Determine the gradient of C at the point P.

The point Q lies on C so the gradient at Q is the same as the gradient at P.

c) Find the coordinates of Q.

$\boxed{}$, $\boxed{(1,-5),(2,-6)}$, $\boxed{36}$, $\boxed{Q(4,22)}$

$$y = 2x^3 - 9x^2 + 12x - 10$$

(a) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\frac{dy}{dx} = 0$

$0 = 6x^2 - 18x + 12$

$0 = x^2 - 3x + 2$

$0 = (x-1)(x-2)$

$x = \begin{cases} 1 \\ 2 \end{cases}$ $\quad y = \begin{cases} 2 - 9 + 12 - 10 = 14 - 19 = -5 \\ 16 - 36 + 24 - 10 = 40 - 46 = -6 \end{cases}$ $\quad \therefore \; (1,-5) \; (2,-6)$

(b) $\frac{dy}{dx} = 6x^2 - 18x + 12$

$\left.\frac{dy}{dx}\right|_{x=-1} = 6(-1)^2 - 18(-1) + 12 = 6 + 18 + 12 = 36$

(c) $\frac{dy}{dx} = 36$

$6x^2 - 18x + 12 = 36$

$6x^2 - 18x - 24 = 0$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$x = \begin{cases} 4 \\ -1 \end{cases}$ ALREADY KNOWN

$y = 2 \times 4^3 - 9 \times 4^2 + 12 \times 4 - 10$

$y = 128 - 144 + 48 - 10$

$y = 176 - 154 = 22$

$\therefore (4, 22)$

Question 17 (***)

The point P, where $x = 2$, lies on the curve with equation

$$y = \frac{1}{6}\left(x^2 + 5\right)^{\frac{3}{2}}, \; x \in \mathbb{R}.$$

Find an equation of the tangent to the curve at P.

$\boxed{}$, $\boxed{y = 3x - \frac{3}{2}}$

$y = \frac{1}{6}(x^2+5)^{\frac{3}{2}}$

$\frac{dy}{dx} = \frac{1}{6} \times \frac{3}{2}(x^2+5)^{\frac{1}{2}} \times 2x$

$\frac{dy}{dx} = \frac{1}{2}x(x^2+5)^{\frac{1}{2}}$

when $x=2$, $y = \frac{9}{2}$, $\left.\frac{dy}{dx}\right|_{x=2} = 3$

$\Rightarrow y - y_0 = m(x - x_0)$

$\Rightarrow y - \frac{9}{2} = 3(x-2)$

$\Rightarrow y - \frac{9}{2} = 3x - 6$

$\Rightarrow y = 3x - \frac{3}{2}$

Question 44 (***)

The equation of the curve C is given by

$$y = e^{2x}(\cos x + \sin x).$$

a) Find an expression for $\frac{dy}{dx}$.

b) Show further that $\frac{dy}{dx}$ can be simplified to

$$\frac{dy}{dx} = e^{2x}(\sin x + 3\cos x).$$

c) Hence show that the x coordinates of the turning points of C satisfy

$$\tan x = -3$$

$\boxed{}$, $\boxed{\frac{dy}{dx} = 2e^{2x}(\cos x + \sin x) + e^{2x}(\cos x - \sin x)}$

(a) $y = e^{2x}(\cos x + \sin x)$

$\Rightarrow \frac{dy}{dx} = 2e^{2x}(\cos x + \sin x) + e^{2x}(-\sin x + \cos x)$

(b) $\frac{dy}{dx} = e^{2x}\left[2\cos x + 2\sin x - \sin x + \cos x\right]$

$\frac{dy}{dx} = e^{2x}\left[3\cos x + \sin x\right]$

(c) For T.P $\frac{dy}{dx} = 0$ $\quad e^{2x} \neq 0$

$\therefore 3\cos x + \sin x = 0$

$\frac{3\cos x}{\cos x} + \frac{\sin x}{\cos x} = \frac{0}{\cos x}$

$3 + \tan x = 0$

$\tan x = -3$

Question 53 (***+)

The curve C has equation

$$y = x^2 e^x, \; x \in \mathbb{R}.$$

a) Find the exact coordinates of the stationary points of C.

b) By considering the sign of $\frac{d^2y}{dx^2}$ at each of these points determine their nature.

$\boxed{}$, $\boxed{\min(0,0)}$, $\boxed{\max\left(-2, \frac{4}{e^2}\right)}$

(a) $y = x^2 e^x$

$\frac{dy}{dx} = 2xe^x + x^2 e^x$

$= xe^x(2+x)$

For MIN/MAX $\frac{dy}{dx} = 0$

$xe^x(2+x) = 0$

$x = 0, -2 \quad (e^x \neq 0)$

$y = 0, 4e^{-2}$

$\therefore (0,0) \; \& \; \left(-2, \frac{4}{e^2}\right)$

(b) $\frac{dy}{dx} = xe^x(x+2)$

$\Rightarrow \frac{dy}{dx} = e^x(x^2+2x)$

$\Rightarrow \frac{d^2y}{dx^2} = e^x(x^2+2x) + e^x(2x+2)$

$\Rightarrow \frac{d^2y}{dx^2} = e^x(x^2+4x+2)$

Hence

$\left.\frac{d^2y}{dx^2}\right|_{x=0} = 2 > 0$ MIN $(0,0)$

$\left.\frac{d^2y}{dx^2}\right|_{x=-2} = -2e^{-2} < 0$ MAX $\left(-2, \frac{4}{e^2}\right)$

Question 54 (***+)

The curve C has equation

$$y = 12\ln x - x^{\frac{3}{2}}, \; x > 0.$$

Determine the range of values of x for which y is decreasing.

$\boxed{x > 4}$

$$y = 12\ln x - x^{\frac{3}{2}}$$

$$\frac{dy}{dx} = \frac{12}{x} - \frac{3}{2}x^{\frac{1}{2}}$$

DECREASING $\Rightarrow \frac{12}{x} - \frac{3}{2}x^{\frac{1}{2}} < 0$

$\Rightarrow \frac{24}{x} - 3x^{\frac{1}{2}} < 0$

$\Rightarrow 24 - 3x^{\frac{3}{2}} < 0$

$\Rightarrow 8 - x^{\frac{3}{2}} < 0$

$\Rightarrow -x^{\frac{3}{2}} < -8$

$\Rightarrow x^{\frac{3}{2}} > 8$

$\Rightarrow x^3 > 64$

$\Rightarrow x > 4$

Question 56 (***+)

The curve C has equation

$$y = \frac{x^2 - 6x + 12}{4x - 11}, \ x \in \mathbb{R}, \ x \neq \frac{11}{4} \ .$$

a) Find a simplified expression for $\frac{dy}{dx}$.

b) Determine the range of values of x, for which y is decreasing.

$\boxed{\frac{dy}{dx} = \frac{4x^2 - 22x + 18}{(4x-11)^2}}$, $\boxed{1 < x < \frac{9}{2}}$

(a) $y = \frac{x^2 - 6x + 12}{4x - 11}$

$\frac{dy}{dx} = \frac{(4x-11)(2x-6) - (x^2-6x+12)\times 4}{(4x-11)^2}$

$\frac{dy}{dx} = \frac{8x^2 - 46x + 66 - 4x^2 + 24x - 48}{(4x-11)^2}$

$\frac{dy}{dx} = \frac{4x^2 - 22x + 18}{(4x-11)^2}$

(b) DECREASING $\Rightarrow \frac{4x^2 - 22x + 18}{(4x-11)^2} < 0$

$\Rightarrow 4x^2 - 22x + 18 < 0$

$\Rightarrow 2x^2 - 11x + 9 < 0$

$\Rightarrow (2x-9)(x-1) < 0$

C.V = 1, $\frac{9}{2}$

$\therefore 1 < x < \frac{9}{2}$

Question 67 (***+)

A curve C has equation

$$y = \frac{4x+k}{4x-k}, \quad x \neq \frac{k}{4},$$

where k is a non zero constant.

a) Find a simplified expression for $\frac{dy}{dx}$, in terms k.

The point P lies on C, where $x = 3$.

b) Given that the gradient at P is $-\frac{8}{27}$, show that one possible value of k is 48 and find the other.

(a) $y = \frac{4x+k}{4x-k}$ $\quad \frac{dy}{dx} = \frac{(4x-k)\times 4 - (4x+k)\times 4}{(4x-k)^2} = \frac{\cancel{16x}-4k-\cancel{16x}-4k}{(4x-k)^2}$

$= \frac{-8k}{(4x-k)^2}$

(b) $\left.\frac{dy}{dx}\right|_{x=3} = -\frac{8}{27}$

$\Rightarrow \frac{-8k}{(12-k)^2} = \frac{-8}{27}$

$\Rightarrow \frac{k}{(12-k)^2} = \frac{1}{27}$

$\Rightarrow 27k = k^2 - 24k + 144$

$\Rightarrow 0 = k^2 - 51k + 144$

$\Rightarrow 0 = (k-48)(k-3)$

$\therefore k = 3$ or 48

Question 76 (***+)

Differentiate each of the following expressions with respect to x, simplifying the final answer as far as possible.

a) $y = \sec^2 x$.

b) $y = x(1-2x)^6$.

c) $y = \frac{\sin x}{2-\cos x}$.

$\boxed{\frac{dy}{dx} = 2\sec^2 x \tan x}$, $\boxed{\frac{dy}{dx} = (14x-1)(2x-1)^5 = (1-14x)(1-2x)^5}$, $\boxed{\frac{dy}{dx} = \frac{2\cos x - 1}{(2-\cos x)^2}$}

(a) $y = \sec^2 x$

$\frac{dy}{dx} = 2\sec x(\sec x \tan x)$

$\frac{dy}{dx} = 2\sec^2 x \tan x$

(b) $y = x(1-2x)^6$

$\frac{dy}{dx} = 1 \times (1-2x)^6 + x \times 6(1-2x)^5(-2)$

$\frac{dy}{dx} = (1-2x)^6 - 12x(1-2x)^5$ OR CONTINUE

$\frac{dy}{dx} = (1-2x)^5\left[(1-2x)-12x\right]$

$\frac{dy}{dx} = (1-2x)^5(1-14x)$

$\frac{dy}{dx} = (2x-1)^5(14x-1)$

(c) $y = \frac{\sin x}{2-\cos x}$

$\frac{dy}{dx} = \frac{(2-\cos x)(\cos x) - \sin x(\sin x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$

$\frac{dy}{dx} = \frac{2\cos x - 1}{(2-\cos x)^2}$

Question 80 (***+)

$$f(x) = \frac{\sin x}{2-\cos x}, \quad 0 \le x < 2\pi \ .$$

a) Find a simplified expression for $f'(x)$.

b) Hence find the minimum and maximum value of $f(x)$.

$\boxed{f(x) = \frac{2\cos x - 1}{(2-\cos x)^2}}$, $\boxed{-\frac{\sqrt{3}}{3} \le f(x) \le \frac{\sqrt{3}}{3}}$

(a) $f(x) = \frac{\sin x}{2-\cos x}$

$\Rightarrow f'(x) = \frac{(2-\cos x)(\cos x) - \sin x(\sin x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - \cos^2 x - \sin^2 x}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - (\cos^2 x + \sin^2 x)}{(2-\cos x)^2}$

$\Rightarrow f'(x) = \frac{2\cos x - 1}{(2-\cos x)^2}$

(b) FOR MIN/MAX $f'(x) = 0$

$2\cos x - 1 = 0$

$\cos x = \frac{1}{2}$

$\arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$

$\left(\begin{array}{l} x = \frac{\pi}{3} \pm 2n\pi \\ x = \frac{5\pi}{3} \pm 2n\pi \end{array} \right.$ $n = 0, 1, 2, 3 \ldots$

$\therefore x_1 = \frac{\pi}{3}$ & $x_2 = \frac{5\pi}{3}$

$\therefore f\left(\frac{\pi}{3}\right) = \frac{\sin \pi/3}{2-\cos \pi/3} = \frac{\sqrt{3}/2}{2-\frac{1}{2}} = \frac{\sqrt{3}}{3}$

$f\left(\frac{5\pi}{3}\right) = \frac{\sin\left(\frac{5\pi}{3}\right)}{2-\cos\frac{5\pi}{3}} = \frac{-\sqrt{3}/2}{2-\frac{1}{2}} = -\frac{\sqrt{3}}{3}$

$\therefore -\frac{\sqrt{3}}{3} \le f(x) \le \frac{\sqrt{3}}{3}$

Question 88 (***+)

The curve C has equation

$$y = \frac{1}{2}x^2 - e^{4x}.$$

Show clearly that C has a point of inflection, and determine its exact coordinates.

$$\boxed{\left[-\ln 2, \frac{1}{2}(\ln 2)^2 - \frac{1}{16}\right]}$$

$$y = \frac{1}{2}x^2 - e^{4x}$$
$$\frac{dy}{dx} = x - 4e^{4x}$$
$$\frac{d^2y}{dx^2} = 1 - 16e^{4x}$$
$$\frac{d^3y}{dx^3} = -64e^{4x}$$

$\frac{d^2y}{dx^2} = 0$ HENCE $1 - 16e^{4x} = 0$

$$1 = 16e^{4x}$$
$$e^{4x} = \frac{1}{16}$$
$$4x = \ln\frac{1}{16}$$
$$4x = -\ln 16$$
$$4x = -4\ln 2$$
$$x = -\ln 2$$

$$y = \frac{1}{2}(-\ln 2)^2 - e^{4(-\ln 2)}$$
$$y = \frac{1}{2}(\ln 2)^2 - \frac{1}{16}$$

$$\left.\frac{d^3y}{dx^3}\right|_{x=-\ln 2} = -64e^{4(-\ln 2)} = -4 \neq 0$$

$\therefore$ POINT OF INFLECTION AT $\left(-\ln 2, \frac{1}{2}(\ln 2)^2 - \frac{1}{16}\right)$

Implicit

Question 11 (***)

A curve has implicit equation

$$y^2 + 3xy - 2x^2 + 17 = 0.$$

Find an equation of the tangent to the curve at the point $(-2,3)$.

$$\boxed{x = -2}$$

$y^2+3xy-2x^2+17=0$

Diff w.r.t x

$2y\frac{dy}{dx}+3y+3x\frac{dy}{dx}-4x=0$

$(2y+3x)\frac{dy}{dx}=4x-3y$

$\frac{dy}{dx}=\frac{4x-3y}{2y+3x}$

$\left.\frac{dy}{dx}\right|_{(-2,3)}=\frac{-8-9}{6-6}=\frac{-17}{0}=\infty$

IE INFINITE GRADIENT $\Rightarrow$ VERTICAL LINE

$\therefore\ x=-2$

Question 12 (***)

The curve C has equation

$$yx(2x-y)+1=0.$$

a) Show clearly that

$$\frac{dy}{dx}=\frac{y^2-4xy}{2x^2-2xy}.$$

The point $P(k,2)$ lies on C.

b) Find the value of k.

c) Show that P is a stationary point of C.

d) Hence, state an equation of the tangent to C at P

$\boxed{}$, $\boxed{k=\frac{1}{2}}$, $\boxed{y=2}$

(a) $yx(2x-y)+1=0$

$2yx^2-y^2x+1=0$

Diff w.r.t x

$2\frac{dy}{dx}x^2+2y(2x)-2y\frac{dy}{dx}x-y^2\times1=0$

$2x^2\frac{dy}{dx}+4xy-2yx\frac{dy}{dx}-y^2=0$

$(2x^2-2xy)\frac{dy}{dx}=y^2-4xy$

$\frac{dy}{dx}=\frac{y^2-4xy}{2x^2-2xy}$ AS REQUIRED

(b) $x=k\quad y=2$

$4k^2-4k+1=0$

$(2k-1)(2k-1)=0$

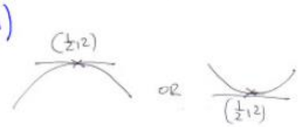
$k=\frac{1}{2}$

(c) $P(\frac{1}{2},2)$

$\left.\frac{dy}{dx}\right|_{(\frac{1}{2},2)}=\frac{2^2-4\times\frac{1}{2}\times2}{2\times(\frac{1}{2})^2-2\times\frac{1}{2}\times2}=\frac{4-4}{\frac{1}{2}-2}=0$

$\therefore$ STATIONARY

(d)



$\therefore\ y=2$

Question 20 (***+)

The point $P(2,3)$ lies on the curve with equation

$$4x^3 - 6xy + 3^y = 23.$$

Show that the gradient of the curve at P is

$$\frac{k}{4 - 9\ln 3},$$

where k is a positive integer to be found.

$\boxed{}$, $\boxed{k = 10}$

$$4x^3 - 6xy + 3^y = 23$$

$$\Rightarrow \frac{d}{dx}(4x^3) - \frac{d}{dx}(6xy) + \frac{d}{dx}(3^y) = \frac{d}{dx}(23)$$

$$\Rightarrow 12x^2 - 6y - 6x\frac{dy}{dx} + 3^y \ln 3 \frac{dy}{dx} = 0$$

AT $(2,3)$

$$\Rightarrow 48 - 18 - 12\frac{dy}{dx}\Big|_{(2,3)} + 27\ln 3 \frac{dy}{dx}\Big|_{(2,3)} = 0$$

$$\Rightarrow (-12 + 27\ln 3)\frac{dy}{dx}\Big|_{(2,3)} = -30$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(2,3)} = \frac{-30}{-12 + 27\ln 3}$$

$$\Rightarrow \frac{dy}{dx}\Big|_{(2,3)} = \frac{10}{4 - 9\ln 3} \quad // \quad \cancel{k} \quad k = 10$$

Question 22 (***+)

A curve C is defined implicitly by

$$y^2 - 3xy + 4x^2 = 28, \quad x \in \mathbb{R}, \; y \in \mathbb{R}.$$

a) Find, in terms of x and y, a simplified expression for $\frac{dy}{dx}$.

b) Determine the coordinates of the stationary points of C.

$\boxed{}$, $\boxed{\frac{dy}{dx} = \frac{3y - 8x}{2y - 3x}}$, $\boxed{(-3,-8),\ (3,8)}$

(a) $y^2 - 3xy + 4x^2 = 28$

Diff w.r.t x

$2y\frac{dy}{dx} - 3y - 3x\frac{dy}{dx} + 8x = 0$

$(2y-3x)\frac{dy}{dx} = 3y - 8x$

$\frac{dy}{dx} = \frac{3y-8x}{2y-3x}$

(b) For T.P $\frac{dy}{dx} = 0$

$3y - 8x = 0$

$\boxed{y = \frac{8}{3}x}$

SOLVING SIMULTANEOUSLY

$\left(\frac{8}{3}x\right)^2 - 3x\left(\frac{8}{3}x\right) + 4x^2 = 28$

$\Rightarrow \frac{64}{9}x^2 - 8x^2 + 4x^2 = 28$

$\Rightarrow 64x^2 - 72x^2 + 36x^2 = 28 \times 9$

$\Rightarrow 28x^2 = 28 \times 9$

$x^2 = \frac{28 \times 9}{28}$

$x^2 = 9$

$x = 3, -3 \qquad y = 8, -8$

$\therefore (3,8)$ & $(-3,-8)$

Rates of change

Question 3 (**)

The volume, V cm^3, of a sphere is given by

$$V = \frac{4}{3}\pi r^3,$$

where r is its radius.

The radius of a sphere is increasing at the constant rate of 2.5 cms^{-1}.

Find the rate at which the volume of the sphere is increasing when its radius is 8 cm.

$\boxed{}$, $\boxed{640\pi \approx 2011 \text{ cm}^3 \text{ s}^{-1}}$

$\boxed{\frac{dr}{dt} = 2.5 \text{ (GIVEN)}}$

$\Rightarrow \frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \times 2.5$

$\Rightarrow \frac{dV}{dt} = 10\pi r^2$

$\Rightarrow \left.\frac{dV}{dt}\right|_{r=8} = 640\pi \approx 2011$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

Question 9 (***)

An oil spillage on the surface of the sea remains circular at all times.

The radius of the spillage, r km, is increasing at the constant rate of 0.5 km h^{-1}.

a) Find the rate at which the area of the spillage, $A \text{ km}^2$, is increasing, when the circle's radius has reached 10 km.

A different oil spillage on the surface of the sea also remains circular at all times.

The area of this spillage, $A \text{ km}^2$, is increasing at the rate of $0.5 \text{ km}^2\text{h}^{-1}$.

b) Show that when the area of the spillage has reached 10 km^2, the rate at which the radius r of the spillage is increasing is

$$\frac{1}{4\sqrt{10\pi}} \text{ km h}^{-1}.$$

$\boxed{10\pi \approx 31.4 \text{ km}^2\text{h}^{-1}}$

(a) $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\Rightarrow \frac{dA}{dt} = (2\pi r) \times 0.5$

$\Rightarrow \frac{dA}{dt} = \pi r$

$\Rightarrow \left.\frac{dA}{dt}\right|_{r=10} = 10\pi$

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

(b) $\frac{dr}{dt} = \frac{dr}{dA} \times \frac{dA}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{2\pi r} \times \frac{1}{2}$

$\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{A=10} = \left.\frac{dr}{dt}\right|_{r=\sqrt{\frac{10}{\pi}}} = \frac{1}{4\pi \times \frac{\sqrt{10}}{\sqrt{\pi}}}$

$\Rightarrow \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{\sqrt{\pi}}{4\pi\sqrt{10}} = \frac{\pi^{\frac{1}{2}}}{4\pi\sqrt{10}}$

$\therefore \left.\frac{dr}{dt}\right|_{r=\frac{\sqrt{10}}{\sqrt{\pi}}} = \frac{1}{4\pi^{\frac{1}{2}}\sqrt{10}} = \frac{1}{4\sqrt{\pi}\sqrt{10}} = \frac{1}{4\sqrt{10\pi}}$ (AS REQUIRED)

$A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$A = \pi r^2$

$10 = \pi r^2$

$\left(\frac{10}{\pi}\right) = r^2$

$r = \sqrt{\frac{10}{\pi}}$

Question 14 (****)

Liquid is pouring into a container at the constant rate of $30\ \text{cm}^3\text{s}^{-1}$.

The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = 36h^2 .$$

a) Find the rate at which the height of the liquid in the container is rising when the height of the liquid reaches 3 cm.

b) Determine the rate at which the height of the liquid in the container is rising 12.5 minutes after the liquid started pouring in.

$\boxed{}$, $\boxed{\frac{5}{36} = 0.139\ \text{cm}\,\text{s}^{-1}}$, $\boxed{\frac{1}{60} = 0.0167\ \text{cm}\,\text{s}^{-1}}$

11 (a) $\boxed{\frac{dV}{dt} = 30}$ GIVEN.

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$\frac{dh}{dt} = \frac{1}{72h} \times 30$$

$$\frac{dh}{dt} = \frac{5}{12h}$$

$$\left.\frac{dh}{dt}\right|_{h=3} = \frac{5}{36} \simeq 0.134$$

$V = 36h^2$

$\frac{dV}{dh} = 72h$

(b) $$\left.\frac{dh}{dt}\right|_{t=12.5\text{min}} = \left.\frac{dh}{dt}\right|_{t=750\text{ seconds}}$$

BUT "CONSTANT RATE" $= 30\ \text{cm}^3/\text{s}$

$\Rightarrow 30 \times 750 = 22500\ \text{cm}^3$

BUT $V = 36h^2$

$\Rightarrow 22500 = 36h^2$

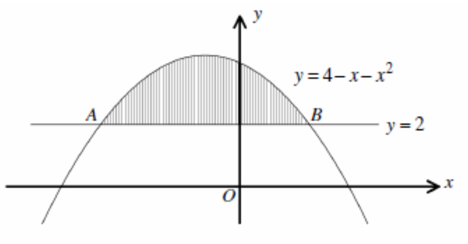
$\Rightarrow h^2 = 625$

$\Rightarrow h = 25$

$$\therefore \left.\frac{dh}{dt}\right|_{h=25} = \frac{5}{12 \times 25} = \frac{1}{60} \simeq 0.016$$

Integration

Question 27 (**+)



The figure above shows a quadratic curve and a straight line with respective equations

$$y = 4 - x - x^2 \quad \text{and} \quad y = 2.$$

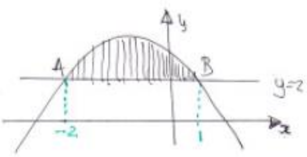
The points A and B are the points of intersection between the quadratic curve and the straight line.

a) Find the coordinates of A and B.

b) Determine the exact area of the finite region bounded by the quadratic curve and the straight line, shown shaded in the above figure.

$\boxed{A(-2,2), B(1,2)}$, $\boxed{\text{area} = \frac{9}{2}}$

(a)



$y = 4 - x - x^2$

$2 = 4 - x - x^2$

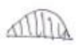
$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1, -2 \quad \therefore \quad A(-2,2), \ B(1,2)$

(b)

$- \boxed{6}$

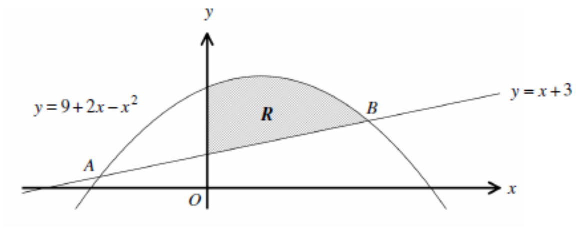


$$\int_{-2}^{1} 4 - x - x^2 \, dx = \left[4x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right]_{-2}^{1} = \left(4 - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(-8 - 2 + \tfrac{8}{3}\right)$$

$$= 4 - \tfrac{1}{2} - \tfrac{1}{3} + 8 + 2 - \tfrac{8}{3} = \tfrac{21}{2}$$

$\therefore$ REQUIRED AREA $= \frac{21}{2} - 6 = \frac{9}{2}$

Question 32 (**+)



The figure above shows the graph of the curve C with equation

$$y = 9 + 2x - x^2,$$

and the straight line L with equation

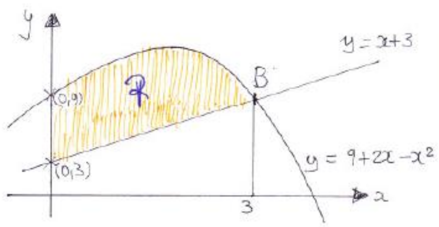
$$y = x + 3.$$

The curve meets the straight line at the points A and B.

The finite region R, shown shaded in the figure, is bounded by the curve C, the straight line L and the coordinate axes.

Show that the area of R is 13.5 square units.

proof



● FIND THE CO-ORDS OF B

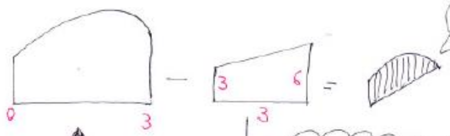
$x+3 = 9+2x-x^2$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = \begin{cases} 3 \leftarrow B \\ -2 \leftarrow A \end{cases}$

$\therefore \boxed{B(3,6)}$



AREA OF TRAPEZIUM

$\frac{3+6}{2} \times 3 = 13.5$

$$\int_0^3 9 + 2x - x^2 \, dx = \left[9x + x^2 - \frac{1}{3}x^3\right]_0^3 = (27 + 9 - 9) - (0) = 27$$

$\therefore$ REQUIRED AREA $= 27 - 13.5 = \frac{27}{2} = 13.5$ // AS REQUIRED

Question 48 (***)

The cubic equation C passes through the origin O and its gradient function is

$$\frac{dy}{dx} = 6x^2 - 6x - 20.$$

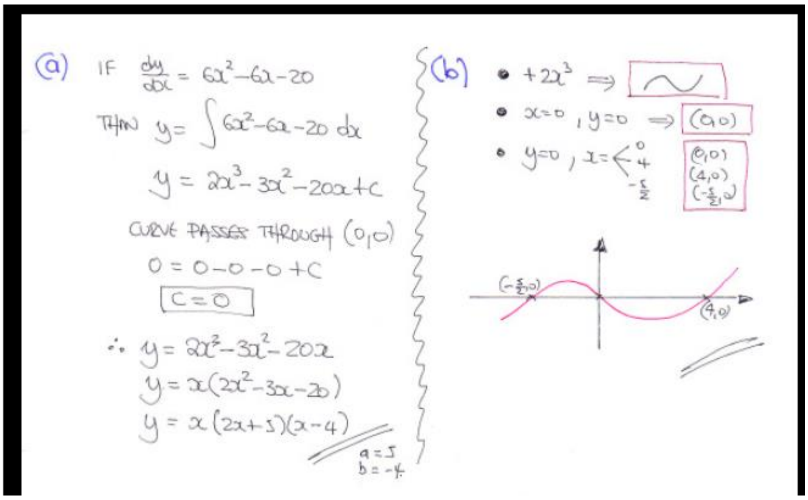
a) Show clearly that the equation of C can be written as

$$y = x(2x+a)(x+b),$$

where a and b are constants.

b) Sketch the graph of C, indicating clearly the coordinates of the points where the graph meets the coordinate axes.

$\boxed{}$, $\boxed{a=5}$, $\boxed{b=-4}$



Question 96 (***+)

The point $P\left(4, \frac{1}{3}\right)$ lies on the curve C whose gradient function is given by

$$\frac{dy}{dx} = \frac{x^{\frac{5}{2}} + 24}{x^2}, \quad x \neq 0.$$

a) Determine an equation of the tangent to C at P.

b) Find an equation of C.

$\boxed{}$, $\boxed{6y = 21x - 82}$, $\boxed{y = \frac{2}{3}x^{\frac{3}{2}} - \frac{24}{x} + 1}$

a) $\boxed{\dfrac{dy}{dx} = \dfrac{x^{\frac{5}{2}}+24}{x^2}}$

- $\left.\dfrac{dy}{dx}\right|_{x=4} = \dfrac{4^{\frac{5}{2}}+24}{4^2} = \dfrac{32+24}{16} = \dfrac{56}{16} = \dfrac{28}{8} = \dfrac{14}{4} = \dfrac{7}{2}$

- EQUATION OF TANGENT? $y - y_0 = m(x - x_0)$

$$y - \frac{1}{3} = \frac{7}{2}(x-4)$$
$$y - \frac{1}{3} = \frac{7}{2}x - 14$$
$$6y - 2 = 21x - 84$$
$$6y = 21x - 82$$

b) $\dfrac{dy}{dx} = \dfrac{x^{\frac{5}{2}}}{x^2} + \dfrac{24}{x^2} = x^{\frac{1}{2}} + 24x^{-2}$

$$y = \int x^{\frac{1}{2}} + 24x^{-2}\,dx = \frac{2}{3}x^{\frac{3}{2}} - 24x^{-1} + C$$

$$\boxed{y = \frac{2}{3}x^{\frac{3}{2}} - \frac{24}{x} + C}$$

AT $\left(4, \frac{1}{3}\right)$

$$\frac{1}{3} = \frac{2}{3}\times 4^{\frac{3}{2}} - \frac{24}{4} + C$$
$$\frac{1}{3} = \frac{16}{3} - 6 + C$$
$$\frac{1}{3} - \frac{16}{3} + 6 = C$$
$$C = 1$$
$$\therefore\ y = \frac{2}{3}x^{\frac{3}{2}} - \frac{24}{x} + 1$$