Sequences and Series Cheat Sheet

A sequence is a list of terms. For example, 3, 6, 9, 12, 15, ...
A series is the sum of a list of terms. For example, 3 + 6 + 9 + 12 + 15 + ...
The terms of a sequence are separated by a comma, while with a series they are all added together.

Definitions

Here are some important definitions prefacing the content in this chapter:

- A sequence is increasing if each term is greater than the previous.
  e.g. 4, 9, 14, 19, ...
- A sequence is decreasing if each term is less than the previous.
  e.g. 5, 4, 3, 2, 1, ...
- A sequence is periodic if the terms repeat in a cycle; $u_{n+k} = u_n$ for some k, which is known as the order of the sequence. e.g. -3, 1, -3, 1, -3, ... is periodic with order 2.

Arithmetic sequences

An arithmetic sequence is one where there is a common difference between each term. Arithmetic sequences are of the form

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \ldots$$

where a is the first term and d is the common difference.

- The n[th] term of an arithmetic series is given by: $\boldsymbol{u_n = a + (n-1)d}$

Factorising out S_n from the LHS and a from the RHS

Arithmetic series

An arithmetic series is the sum of the terms of an arithmetic sequence.

- The sum of the first n terms of an arithmetic series is given by $\boldsymbol{S_n = \frac{n}{2}[2a + (n-1)d]}$ or $\boldsymbol{S_n = \frac{n}{2}(a + l)}$

where a is the first term, d is the common difference and l is the last term.

You need to be able to prove this result. Here is the proof:

Example 1: Prove that the sum of the first n terms of an arithmetic series is $S_n = \frac{n}{2}[2a + (n-1)d]$.

We start by writing the sum out normally [1], and then in reverse [2]:

[1] $S_n = a + (a+d) + (a+2d) + \cdots + (a + (n-2)d) + (a + (n-1)d)$

[2] $S_n = (a + (n-1)d) + (a + (n-2)d) + \cdots + (a+2d) + (a+d) + a$

Adding [1] and [2] gives us:

[1] + [2]: $2S_n = n(2a + (n-1)d)$

$$\therefore S_n = \frac{n}{2}[2a + (n-1)d]$$

Example 2: The fifth term of an arithmetic series is 33. The tenth term is 68. The sum of the first n terms is 2225. Show that $7n^2 + 3n - 4450 = 0$, and hence find the value of n.

$5th\ term\ is\ 33 \therefore a + 4d = 33$ [1] (n[th] term formula)

$10th\ term\ is\ 68 \therefore 68 = a + 9d$ [2] (n[th] term formula)

Solving [1] and [2] simultaneously, we find that $d = 7$ and $a = 5$.

Sum of first n terms is 2225 $\therefore \frac{n}{2}[2a + (n-1)d] = 2225$

$$\frac{n}{2}[10 + (n-1)(7)] = 2225$$

$$n(3 + 7n) = 2225(2)$$

$$7n^2 + 3n - 4450 = 0$$

To find the value of n, we just need to solve the quadratic. Using the quadratic formula, we find that $n = 25\ or\ n = -25.4$. Since the term number must be a positive integer, we can conclude that $n = 25$.

Geometric sequences

The defining feature of a geometric sequence is that you must multiply by a common ratio, r, to get from one term to the next. Geometric sequences are of the form

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad ar^4, \quad \ldots$$

where a is the first term in the sequence and r is the common ratio.

- The n[th] term of a geometric sequence is given by: $\boldsymbol{u_n = ar^{n-1}}$

It can help in many questions to use the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$. This is especially helpful when the terms of the sequence are given in terms of an unknown constant. Part a of example 4 highlights this.

Geometric series

A geometric series is the sum of the terms of a geometric sequence.

- The sum of the first n terms of a geometric series is given by:

$$\boldsymbol{S_n = \frac{a(1-r^n)}{1-r}}$$

by multiplying the top and bottom of the fraction by -1, we can also use

$$\boldsymbol{S_n = \frac{a(r^n - 1)}{r-1}}$$

You need to be able to prove this result. Here is the proof:

Example 3: Prove that the sum of the first n terms of a geometric series is $S_n = \frac{a(1-r^n)}{1-r}$

$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$ [1]

multiplying the sum by r $\quad rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$ [2]

Subtracting [2] from [1] $\quad S_n - rS_n = a - ar^n$

$\Rightarrow S_n(1-r) = a(1-r^n)$ Factoring out S_n and a

$\therefore S_n = \frac{a(1-r^n)}{1-r}$ Dividing by $1-r$

Since division by zero is undefined, this formula is invalid when $r = 1$.

Sum to infinity

The sum to infinity of a geometric sequence is the sum of the first n terms as n approaches infinity. This does not exist for all geometric sequences. Let's look at two examples:

$$2 + 4 + 8 + 16 + 32 + \cdots$$

Each term is twice the previous (i.e. $r = 2$). The sum of such a series is not finite, since each term is bigger than the previous. This is known as a divergent sequence.

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Here, each term is half the previous (i.e. $r = \frac{1}{2}$). The sum of such a series is finite, since as n becomes large, the terms will tend to 0. This is known as a convergent sequence.

- A geometric sequence is convergent if and only if $|r| < 1$.

The sum to infinity of a geometric sequence only exists for convergent sequences, and is given by:

$$\boldsymbol{S_\infty = \frac{a}{1-r}}$$

Example 4: The first three terms of a geometric series are (k-6), k, (2k+5), where k is a positive constant.

a) Show that $k^2 - 7k - 30 = 0$
b) Hence find the value of k.
c) Find the common ratio of this series and hence calculate the sum of the first 10 terms.
d) Find the sum to infinity for a series with first term k and common ratio $\frac{k-6}{k}$.

a) Using the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$	$\frac{k}{k-6} = \frac{2k+5}{k}$
Cross-multiplying and simplifying:	$\Rightarrow k^2 = (2k+5)(k-6)$ $\Rightarrow k^2 = 2k^2 - 7k - 30$ $\Rightarrow k^2 - 7k - 30 = 0$ as required.
b) Solving the quadratic:	$(k-10)(k+3) = 0 \quad \Rightarrow \quad k = 10, \quad k = -3$ Since we are told k is positive, we can conclude $k = 10$.
c) From part a, $\frac{u_{k+1}}{u_k} = r = \frac{10}{10-6} = \frac{5}{2}$	$S_{10} = \frac{a(1-r^{10})}{1-r} = \frac{4(1-(2.5)^{10})}{1-2.5} = 25428.6 \Rightarrow 25400\ to\ 3\ s.f.$
d) $a = 10\ and\ r = \frac{1}{\frac{5}{2}} = \frac{2}{5}$	$\therefore S_\infty = \frac{10}{1-\frac{2}{5}} = \frac{50}{3}$

Recurrence relations

A recurrence relation is simply another way of defining a sequence. With recurrence relations, each term is given as a function of the previous. For example, $u_{n+1} = u_n + 4$, $u_1 = 1$ represents an arithmetic sequence with first term 1 and common difference 4. In order to generate a recurrence relation, you need to know the first term.

Example 5: The sequence with recurrence relation $u_{k+1} = pu_k + q$, $u_1 = 5$, where p is a constant and $q = 13$, is periodic with order 2. Find the value of p.

We know the order is 2. So if $u_1 = 5$, then $u_3 = 5$ too. Finding u_3:	$u_2 = pu_1 + 13 = 5p + 13$ $u_3 = pu_2 + 13 = p(5p+13) + 13$
Equating to 5:	$p(5p+13) + 13 = 5$
Simplifying:	$5p^2 + 13p + 8 = 0$
Solving the quadratic by factorising: We get 2 values, one of which is correct.	$(5p+8)(p+1) = 0$ $p = -1$ or $p = -1.6$
Substitute $p = -1.6$ and $p = -1$ into the recurrence relation separately to see which one correctly corresponds to a periodic sequence of order 2.	Substituting $p = -1.6$ into the recurrence relation gives a sequence where each term is 5 and so does not have order 2. Using $p = -1$ does give us a periodic sequence with order 2 however, so $p = -1$.

Sigma notation

You need to be comfortable solving problems where series are given in sigma notation. Below is an annotated example explaining how the sigma notation is used.

This tells us the last value of r for our sequence, i.e. our last term will be $7 - 2(20) = -33$

Inputting $r = 1$ into this expression gives us the first term, $r = 2$ gives us the second term, and so forth.

$$\sum_{r=1}^{20}(7 - 2r) = 5 + 3 + 1 - 1 + \cdots$$

This is the value of r where our series starts, i.e. our first term is $7 - 2(1) = 5$.

This series in particular represents an arithmetic series with first term 5 and common difference -2

If you are ever troubled by a series given in sigma notation, it is a good idea to write out the first few terms and analyse the series that way.

Modelling with series

Geometric and arithmetic sequences are often used to model real-life scenarios. Consider the amount of money in a savings account; this can be modelled by a geometric sequence where r represents the interest paid at the end of each year and a is the amount of money in the account at the time of opening.

You need to be able to apply your knowledge of sequences and series to questions involving real-life scenarios. It is important to properly understand the context given to you, so take some time to read through the question more than once.

Example 6: A virus is spreading such that the number of people infected increases by 4% each day. Initially 100 people were diagnosed with the virus. How many days will it be before 1000 are infected?

This is a geometric series with $a = 100$ and $r = 1.04$. We are really just trying to find the smallest value of n such that $S_n > 1000$.	$S_n = \frac{100(1-1.4^n)}{1-1.4} > 1000$
the sign flips as we multiply by a negative number (-0.4)	$100(1 - 1.04^n) < -400$
Dividing through by 100:	$1 - 1.4^n < \frac{-400}{100}$ $1.4^n > 5$
Taking logs of both sides:	$\log(1.4^n) > \log(5)$
Using the power rule for logs:	$nlog(1.4) > \log(5)$
Dividing through by log 1.4. We round our answer up.	$\therefore n > \frac{log(5)}{\log(1.4)} = 4.78\ \ so\ n = 5$