Sequences and Series Cheat Sheet

A sequence is a list of terms. For example, 3, 6, 9, 12, 15, ... A series is the sum of a list of terms. For example, 3 + 6 + 9 + 12 + 15 + ... The terms of a sequence are separated by a comma, while with a series they are all added together.

Definitions

Here are some important definitions prefacing the content in this chapter:

- A sequence is increasing if each term is greater than the previous. e.g. 4, 9, 14, 19. ...
- A sequence is decreasing if each term is less than the previous. e.g. 5. 4. 3. 2. 1. ..
- A sequence is periodic if the terms repeat in a cycle; $u_{n+k} = u_n$ for some k, which is known as the order of the sequence. e.g. -3, 1, -3, 1, -3, ... is periodic with order 2.

Arithmetic sequences

An arithmetic sequence is one where there is a common difference between each term. Arithmetic sequences are of the form

 $a, a+d, a+2d, a+3d, \dots$

where a is the first term and d is the common difference.

• The nth term of an arithmetic series is given by: $u_n = a + (n-1)d$

Arithmetic series

Factorising out S_n from the LHS and *a* from the RHS

An arithmetic series is the sum of the terms of an arithmetic sequence

• The sum of the first *n* terms of an arithmetic series is given by $S_n = \frac{n}{2} [2a + (n-1)d]$ or $S_n = \frac{n}{2}(a+l)$

where a is the first term, d is the common difference and l is the last term.

You need to be able to prove this result. Here is the proof:

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Example 1: Prove that the sum of the first n terms of an arithmetic series is S_n = \frac{n}{2} [2a + (n-1)d].
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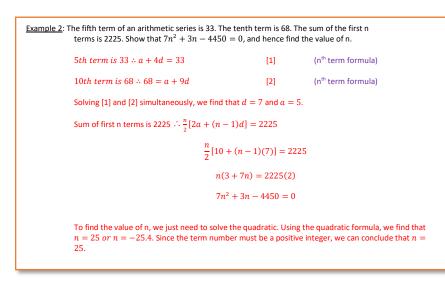
We start by writing the sum out normally [1], and then in reverse [2]:

- [1] $S_n = a + (a + d) + (a + 2d) + \dots + (a + (n 2)d) + (a + (n 1)d)$
- [2] $S_n = (a + (n-1)d) + (a + (n-2)d) + \dots + (a+2d) + (a+d) + a$

Adding [1] and [2] gives us:

 $[1] + [2]: \qquad 2S_n = n(2a + (n-1)d)$

 $\therefore S_n = \frac{n}{2} [2a + (n-1)d]$





Geometric sequences

The defining feature of a geometric sequence is that you must multiply by a common ratio, r, to get from one term to the next. Geometric sequences are of the form

 $a, ar, ar^2, ar^3, ar^4, ...$

where a is the first term in the sequence and r is the common ratio.

• The nth term of a geometric sequence is given by: $u_n = ar^{n-1}$

It can help in many questions to use the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$. This is especially helpful when the terms of the sequence are given in terms of an unknown constant. Part a of example 4 highlights this.

Geometric series

A geometric series is the sum of the terms of a geometric sequence.

• The sum of the first n terms of a geometric series is given by:

$$S_n = \frac{a(1-r^n)}{1-r}$$

by multiplying the top and bottom of the fraction by -1, we can also use

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

You need to be able to prove this result. Here is the proof:

Example 3: Prove that the sun	n of the first n terms of a geometric series is $S_n =$	$\frac{a(1-r^n)}{1-r}$
	$S_n = a + ar + ar^2 + \dots + ar^{n-1}$	[1]
multiplying the sum by r	$rS_n = ar + ar^2 + ar^3 + \dots + ar^n$	[2]
Subtracting [2] from [1]	$S_n - rS_n = a - ar^n$	
	$\Rightarrow S_n(1-r) = a(1-r^n)$	Factoring out S_n and a
	$\therefore S_n = \frac{a(1-r^n)}{1-r}$	Dividing by $1-r$

Since division by zero is undefined, this formula is invalid when r = 1.

Sum to infinity

The sum to infinity of a geometric sequence is the sum of the first n terms as n approaches infinity. This does not exist for all geometric sequences. Let's look at two examples:

 $2 + 4 + 8 + 16 + 32 + \cdots$

Each term is twice the previous (i.e. r = 2). The sum of such a series is not finite, since each term is bigger than the previous. This is known as a divergent sequence.

$$2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$$

Here, each term is half the previous (i.e. $r = \frac{1}{r}$). The sum of such a series is finite, since as n becomes large, the terms will tend to 0. This is known as a convergent sequence

• A geometric sequence is convergent if and only if |r| < 1.

The sum to infinity of a geometric sequence only exists for convergent sequences, and is given by:

$$S_{\infty}=\frac{a}{1-r}$$

0

a) Show that
$$k^2 - 7k - 30$$

b) Hence find the value of k
c) Find the common ratio of
d) Find the sum to infinity for
a) Using the fact that $\frac{u_{k+1}}{u_k} = \frac{u_{k+2}}{u_{k+1}} = r$
Cross-multiplying and simplifying:
b) Solving the quadratic:
c) From part a, $\frac{u_{k+1}}{u_k} = r = \frac{10}{10-6} = \frac{5}{2}$
d) $a = 10$ and $r = \frac{1}{\frac{5}{2}} = \frac{2}{5}$

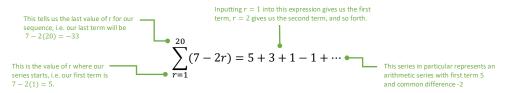
Recurrence relations

4. In order to generate a recurrence relation, you need to know the first term.

Example 5:The sequence with recurrentorder 2. Find the value of
$$p$$
.We know the order is 2. So if $u_1 = 5$, totoo. Finding u_3 :Equating to 5:Simplifying:Solving the quadratic by factorising:We get 2 values, one of which is correetSubstitute $p = -1.6$ and $p = -1$ intorecurrence relation separately to see a correctly corresponds to a periodic second correct.

Sigma notation

how the sigma notation is used.



that way.

Modelling with series

Geometric and arithmetic sequences are often used to model real-life scenarios. Consider the amount of money in a savings account; this can be modelled by a geometric sequence where r represents the interest paid at the end of each year and a is the amount of money in the account at the time of opening.

You need to be able to apply your knowledge of sequences and series to questions involving real-life scenarios. It is important to properly understand the context given to you, so take some time to read through the question more than once.

Example 6: A virus is spreading such that diagnosed with the virus. How many day

This is a geometric series with a = 100We are really just trying to find the sm such that $S_n > 1000$ the sign flips as we multiply by a negat Dividing through by 100:

Taking logs of both sides:

Using the power rule for logs: Dividing through by log 1.4. We round

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Edexcel Pure Year 2 Example 4: The first three terms of a geometric series are (k-6), k, (2k+5), where k is a positive constant. = 0f this series and hence calculate the sum of the first 10 terms. or a series with first term k and common ratio $\frac{k-6}{k}$. 2k + 5 $\frac{1}{k-6} = \frac{1}{k}$ $\Rightarrow k^2 = (2k+5)(k-6)$ $\Rightarrow k^{2} = 2k^{2} - 7k - 30$ $\Rightarrow k^{2} - 7k - 30 = 0 \text{ as required}$ $(k-10)(k+3) = 0 \implies k = 10, \quad k = -3$ Since we are told k is positive, we can conclude k = 10. $S_{10} = \frac{a(1-r^{10})}{r} = \frac{4(1-(2.5)^{10})}{r} = 25428.6 \Rightarrow 25400 \text{ to } 3 \text{ s. } f.$ $S_{10} = \frac{1}{1-r} = \frac{1}{1-2.5}$ $\therefore S_{\infty} = \frac{10}{1-\frac{2}{2}} = \frac{50}{3}$ $1 - \frac{2}{5}$

A recurrence relation is simply another way of defining a sequence. With recurrence relations, each term is given as a function of the previous. For example, $u_{n+1} = u_n + 4$, $u_1 = 1$ represents an arithmetic sequence with first term 1 and common difference

nce relation u_{k+1}	$p_{k+1} = pu_k + q$, $u_1 = 5$, where p is a constant and $q = 13$, is periodic with
then $u_3 = 5$	$u_2 = pu_1 + 13 = 5p + 13$
	$u_3 = pu_2 + 13 = p(5p + 13) + 13$
	p(5p+13) + 13 = 5
	$5p^2 + 13p + 8 = 0$
	(5p+8)(p+1) = 0
ect.	p = -1 or $p = -1.6$
o the	Substituting $p = -1.6$ into the recurrence relation gives a
which one	sequence where each term is 5 and so does not have order 2.
equence of	Using $p = -1$ does give us a periodic sequence with order 2
	however, so $p = -1$.

You need to be comfortable solving problems where series are given in sigma notation. Below is an annotated example explaining

If you are ever troubled by a series given in sigma notation, it is a good idea to write out the first few terms and analyse the series

ays will it be before 1000 are infected?			
00 and $r = 1.04$. mallest value of n	$S_n = \frac{100(1 - 1.4^n)}{1 - 1.4} > 1000$		
ative number (-0.4)	$100(1-1.04^n) < -400$		
	$1 - 1.4^n < \frac{-400}{100}$		
	$1.4^n > 5$		
	$\log(1.4^n) > \log(5)$		
	nlog(1.4) > log(5)		
d our answer up.	$\therefore n > \frac{\log(5)}{\log(1.4)} = 4.78 \text{ so } n = 5$		

