Integration Cheat Sheet

Edexcel Pure Year 2

Integration is the inverse of differentiation. We can think of integration as a mathematical tool that allows us to find areas enclosed between curves and the coordinate axes. In fact, the uses of integration extend far beyond finding areas and can also be found in other fields of study, including Physics, Statistics and even Economics.

Very useful results

$f(x)$	$\int f(x)\,dx$
x^n	$\frac{x^{n+1}}{n+1}$
e^{ax+b}	$\frac{1}{a}e^{ax+b}$
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$
$(ax+b)^n$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$
$\sin kx$	$-\frac{1}{k}\cos kx$
$\cos kx$	$\frac{1}{k}\sin kx$

Results you will be given

$f(x)$	$\int f(x)\,dx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\operatorname{cosec} kx$	$-\frac{1}{k}\ln\lvert\operatorname{cosec} kx+\cot kx\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx+\tan kx\rvert$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\sec kx\tan kx$	$\frac{1}{k}\sec kx$
$\operatorname{cosec}^2 kx$	$-\frac{1}{k}\cot kx$
$\operatorname{cosec} kx\cot kx$	$-\frac{1}{k}\operatorname{cosec} kx$

These can be derived from the differentiation section in the formula booklet

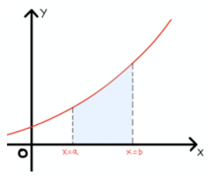
"by parts" formula: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

Trapezium rule: $\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\}$, where $h=\frac{b-a}{n}$

You will also be given these in the formula booklet.

Finding Areas

Just like in Pure Year 1, you will need to use integration to find areas. You should remember:



- $\int_a^b y\,dx$ represents the area bounded between the curve $y=f(x)$, the x-axis and the lines $x=a$ and $x=b$.
- When a function is given parametrically, the area under the curve is given by $\int_a^b \left[y\frac{dx}{dt}\right]dt$. Remember that a and b are limits given in terms of t, and the integration is done with respect to t.
- You need to be able to recognise that $\lim_{\delta x\to 0}\sum_{x=a}^{b} f(x)\delta x = \int_a^b f(x)\,dx$

Solving integrals involving trigonometric equations

You might come across an expression involving trigonometric functions that you can't integrate straight away using one of the results above. For example, $\int \sin^3 x\,dx$ can't be found directly.

To tackle problems of this sort, you need to manipulate the expression into a form you can integrate. This is why it is crucial you are familiar with all of the identities you encountered in chapters 6 and 7. Below is an example showcasing we do this in practice.

Example 1: Evaluate $\int \sin^2 x\,dx$

$\int \sin^2 x\,dx = \int (1-\cos^2 x)dx = \int 1\,dx - \int \cos^2 x\,dx.$ — using $\sin^2 x = 1-\cos^2 x$

$But\ \cos 2x = 2\cos^2 x - 1, \quad so\ \cos^2 x = \frac{\cos 2x + 1}{2}$

$\therefore \int \cos^2 x\,dx = \frac{1}{2}\int \cos 2x + 1\,dx = \frac{1}{2}\left[\frac{1}{2}\sin 2x + x\right] = \frac{1}{4}\sin 2x + \frac{1}{2}x$ — using the double angle formulae $\cos 2x = 2\cos^2 x - 1$

$so\ \int \sin^2 x\,dx = x - \frac{1}{4}\sin 2x - \frac{1}{2}x + c$

Reverse Chain Rule

Some complicated expressions can be integrated very easily if they are of one of the forms below:

[1] $\int f'(x)[f(x)]^n\,dx = \frac{f(x)^{n+1}}{n+1} + c.$

e.g. $\int \sin^{400}x\cos x\,dx = \frac{\sin^{401}x}{401} + c$

$\int x^2\sqrt{x^3+4}\,dx = \frac{1}{3}\cdot\frac{(x^3+4)^{\frac{3}{2}}}{\frac{3}{2}}$

[2] $\int \frac{f'(x)}{f(x)}dx = \ln\lvert f(x)\rvert + c$

e.g. $\int \frac{2x}{x^2+42}dx = \ln\lvert x^2+42\rvert + c$

$\int \frac{\sin x\cos x}{2\cos 2x+1}dx = \int \frac{\frac{1}{2}\sin 2x}{2\cos 2x+1}dx = \frac{1}{2}\left[-\frac{1}{2}\ln\lvert 2\cos 2x+1\rvert\right] + c$

Remember that you must adjust for any variation in constants. In example 2 of rule 1, we had to multiply our answer by $\frac{1}{3}$ since the differential of $(x^3+4)^5$ is $3x^2$, not x^2.

While it is true you do not necessarily need to know the above rules, it is still very worthwhile for you to take the time to learn to apply them because they can greatly simplify otherwise difficult integrals.

Using partial fractions

You could also be asked to integrate an expression involving a fraction with more than one linear factor in the denominator. Take for example, the expression $\frac{11x^2+14x+5}{(x+1)^2(2x+1)}$. This cannot be integrated directly so we need to simplify it, which is where partial fractions come in handy. Using partial fractions, we can rewrite this as $\frac{4}{x+1} - \frac{2}{(x+1)^2} + \frac{3}{(2x+1)}$.

Now, each of the above terms can be integrated directly using one of the standard results on the left. Carrying out the integration:

$\int \frac{11x^2+14x+5}{(x+1)^2(2x+1)}dx = \int \frac{4}{x+1} - \frac{2}{(x+1)^2} + \frac{3}{(2x+1)}dx$

$= 4\ln\lvert x+1\rvert + \frac{2}{x+1} + \frac{3}{2}\ln\lvert 2x+1\rvert + c$

Such questions are often worth upwards of 6 marks and are heavily reliant on your ability to use partial fractions well. The integration is mostly straight forward and does not require much extra work outside of applying the standard results, but it is the partial fractions procedure that can be a little tedious.

Trapezium Rule

In some cases, you might not be able to integrate a function algebraically. We can instead use a numerical method called the trapezium rule to find an estimate for the area under the curve.

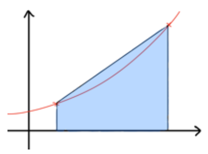
The concept is simple: we divide the area required into vertical 'strips' which each form trapezia, find the approximate area of each strip (using the area of a trapezium formula) and finally add them all up giving us the total area. The more strips we use with the trapezium rule, the more accurate our estimate is. The formula for the trapezium rule is:

$$\int_a^b y\,dx \approx \frac{1}{2}h\{(y_0+y_n)+2(y_1+y_2+\ldots+y_{n-1})\},\ \text{where } h=\frac{b-a}{n}$$

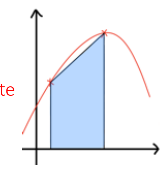
Y-value at starting point; Y-value at end point; End x-value; Starting x-value; Number of strips used

You could also be asked to determine whether your estimate is an overestimate or an underestimate. To do so, you must look at the graph of the curve in the region you are estimating and determine whether it is concave or convex.

Convex, Overestimate



Concave, Underestimate



Another common question is to find the percentage error of your estimate. To find this, you can use the formula for percentage error:

$$\%\ \text{error} = \frac{estimate - exact}{exact}\times 100$$

Here, "exact" represents the exact value of the integral, which you would need to find algebraically.

The Substitution Method

Integration via substitution is a powerful technique that can be used to solve more complex integrals, which might not be solvable using methods we have looked at so far.

The idea is that we can pick a new variable, often called "u", which replaces the existing variable we had in an attempt to simplify the integral. This is best illustrated by an example:

Example 3: Find $\int_6^{20}\frac{8x}{\sqrt{4x+1}}dx$ using the substitution $u^2 = 4x+1$.	
[1] Differentiate $u^2 = 4x+1$ to find dx in terms of du: You will need to use implicit differentiation here. You could instead make u the subject by taking the square root of both sides of the equation before differentiating, but the differentiation becomes messy and less convenient.	$2u\frac{du}{dx} = 4 \quad \therefore \quad \frac{dx}{du} = \frac{u}{2}$ so we can say $dx = \frac{u}{2}du$
[2] We now need to substitute out all of the x terms. Using $u^2 = 4x+1$, we can see that $8x = 2u^2 - 2$. We also replace dx with $\frac{u}{2}du$:	$\int \frac{2u^2-2}{\sqrt{u^2}}\cdot\frac{u}{2}du = \int u^2 - 1\,du$
[3] We now need to find our new limits. Again, we can use our substitution $u^2 = 4x+1$ to do this:	At $x=6, u=\sqrt{4(6)+1}=\sqrt{25}=5$ At $x=20, u=\sqrt{4(20)+1}=\sqrt{81}=9$
[4] Finally, our "x" integral has been completely transformed into a "u" integral so we can proceed to evaluate:	$\int_5^9 u^2 - 1\,du = \left[\frac{u^3}{3} - u\right]_5^9 = [234] - \left[\frac{110}{3}\right] = \frac{592}{3}$

Here are some helpful pointers regarding the substitution method:

- In the exam, you will often be told which substitution to use. If not, then a good rule of thumb is to "try whatever is inside brackets or square root". For example, if you are given $\int x\sqrt{3x+4}\,dx$, a good choice of substitution would be $u = 3x+4$. Be aware that this "rule" may not always work, but it is helpful to try if you are unsure what to substitute.
- You might occasionally find it difficult to spot a substitution that would work in hindsight. In this case, it is best to simply try a couple and see whether they are helpful or not.
- If you decide to use a substitution when evaluating an indefinite integral, don't forget to give your final answer in terms of the variable you started with!

Integration by Parts

When we want to integrate an expression that is a product of two functions, we can use integration by parts. Think back to the product rule for differentiation; the idea is the same. The formula is as follows:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

or, if we are using limits:

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b v\frac{du}{dx}dx$$

When using this method, we need to pick one of our functions to be $\frac{dv}{dx}$ and the other to be u. This choice is crucial and should be made based on the "LATE" rule:

Easier to integrate →

	L	A	T	E
	Logarithms	Algebra	Trig.	Exponentials
e.g	$\ln 2x$.	$2x^2+4$	$6\sin(4x)$	e^{3x}

Whichever function in your expression is easier to integrate should be selected as $\frac{dv}{dx}$. Once you have made this choice, you can proceed to using the formula. Here are some key points to keep in mind when integrating by parts:

- With some questions you may need to apply the 'by parts' formula more than once to get to the final answer. An example of this would be if you were asked to evaluate $\int x^2e^x\,dx$.
- When evaluating $\int \ln x\,dx$ in particular, you need to use 'by parts". To do so, let $\frac{dv}{dx} = 1$.

Solving Differential Equations

Any equation involving derivatives is known as a differential equation. The order of a differential equation is the order of the highest derivative in the equation. You need to be able to solve differential equations of first order, using a method known as Separation of Variables:

When $\frac{dy}{dx} = f(x)g(y)$, we can write $\int \frac{1}{g(y)}dy = \int f(x)\,dx$

Keep in mind that once we carry out the integration, there will be an unknown constant of integration left over. This is why the solution that we attain is known as a general solution. Sometimes, you will be given a boundary condition, which tells you a point that your solution passes through. This allows you to find a particular solution. Remember: the solution to a differential equation is a function!

The process is always the same when it comes to solving differential equations and can be summarised as:

[1] Rearrange the equation you are given into the form $\frac{dy}{dx} = f(x)g(y)$.

[2] Evaluate $\int \frac{1}{g(y)}dy = \int f(x)\,dx$, using your knowledge of integration.

[3] Add the constant of integration and rearrange your general solution into the required form (this is given to you in the question).

[4] If you need to find the particular solution, substitute the given boundary condition into your general solution and find your unknown constant.