Numerical methods Cheat Sheet

Edexcel Pure Year 2

Some mathematical equations that form in the real world turn out to be very difficult to solve; finding an exact solution is either very time consuming or impossible using techniques we already know. Take for example, the equation $3ln|2x^2| + 4cosx - e^x = 0$. This cannot be solved using any techniques you have learnt so far. We can instead use numerical methods to find approximations to the solutions of such equations.

Locating roots

A root of a function is a value of x for which $f(x) = 0$. In other words, a root is where $f(x)$ crosses the x-axis.

- If $f(x)$ is continuous on the interval $[a, b]$ and $f(a)$ has an opposite sign to $f(b)$, then $f(x)$ has at least one root in this interval.

When we say $f(x)$ has to be continuous on an interval, this just means that when graphed, the function is unbroken. In other words, you could trace the function with a pen without needing to lift your pen off the paper.

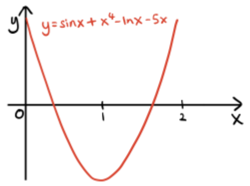
Let's look at an example to clarify why the above bullet point makes sense:

<u>Example 1:</u> Show, without using the graph, that the function $y = sinx + x^4 - lnx - 5x$ has a root α in the interval $[1, 2]$.

Calculating $f(1)$: $sin1 + 1^4 - ln1 - 5 = -3.1585$ ...
Calculating $f(2)$: $sin2 + 2^4 - ln2 - 5(2) = 6.216$ ...
We can see a change of sign occurs between $x = 1$ and $x = 2$.

This means that somewhere between $x = 1$ and $x = 2$, the function crosses the x-axis. Hence, a root must lie in the mentioned interval.

This is confirmed by the graph, which clearly shows the curve crossing the x-axis in the same interval.

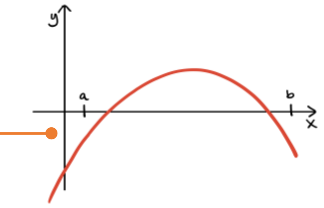


Changes of sign and roots

If there isn't a change of sign in an interval, that does not necessarily mean a root does not exist. Also, if there is a change of sign, that does not necessarily mean only one root exists. There are three cases you need to be wary of:

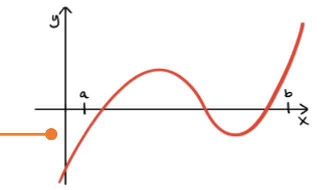
Case 1: Multiple roots with no sign change
If there is more than one root in an interval, $f(a)$ and $f(b)$ may have the same sign.

Here, there are two roots between a and b but $f(a)$ and $f(b)$ are both negative.



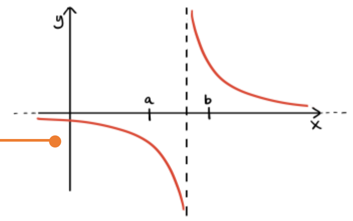
Case 2: Multiple roots with a sign change
There can be more than one root present if there is a change in sign.

There are three roots between a and b and $f(a)$ and $f(b)$ have opposing signs.



Case 3: Vertical asymptotes
When vertical asymptotes are present, a sign change will occur without there being any root.

The graph never crosses the x axis yet $f(a)$ and $f(b)$ have opposing signs.



Using iteration

If we wish to find the roots of an equation $f(x) = 0$, we can use an iterative method.

1. To solve an equation of the form $f(x) = 0$, rearrange the equation into the form $x = g(x)$ and use the iterative formula $x_{n+1} = g(x_n)$.

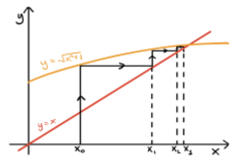
You must be careful when using an iterative method as not all iterations will converge to a root. Sometimes, successive iterations will move away from the root quickly. This is known as divergence.

<u>Example 2:</u> $f(x) = x^3 - 3x^2 - 2x + 5$
a) Show that $f(x) = 0$ has a root α in the interval $3 < x < 4$,
b) Show that the equation $f(x) = 0$ can be written as $x = \sqrt{\frac{x^3-2x+5}{3}}$.
c) Use the iterative formula $x_{n+1} = g(x_n)$ to find the value of x_1, x_2 and x_3, with (i) with $x_0 = 1.5$, (ii) with $x_0 = 4$.

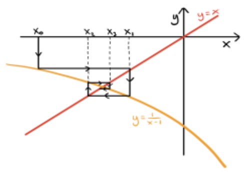
a) We must calculate $f(3)$ and $f(4)$ and show that there is a change in sign.	$f(3) = 3^3 - 3(3^2) - 2(3) + 5 = -1$ $f(4) = 4^3 - 3(4^2) - 2(4) + 5 = 13$ As there is a change in sign between $x = 3$ and $x = 4$, this proves that a root lies in this interval.
b) With such questions, the clue is in what you want to show. The expression is square rooted, which tells us we want to first make x^2 the subject.	$f(x) = 0 \Rightarrow x^3 - 3x^2 - 2x + 5 = 0$
Making x^2 the subject.	$\Rightarrow 3x^2 = x^3 - 2x + 5$ $\Rightarrow x^2 = \frac{x^3 - 2x + 5}{3}$
Square rooting:	$\therefore x = \sqrt{\frac{x^3 - 2x + 5}{3}}$
c) The iterative formula we need to use is $x_{n+1} = \sqrt{\frac{{x_n}^3-2x_n+5}{3}}$.	
$(i)\ x_0 = 1.5$	$(ii)\ x_0 = 4$
$x_1 = \sqrt{\frac{1.5^3-2(1.5)+5}{3}} = 1.3385..$	$x_1 = \sqrt{\frac{4^3-2(4)+5}{3}} = 4.5092..$
$x_2 = \sqrt{\frac{1.3385..^3-2(1.3385..)+5}{3}} = 1.2544..$	$x_2 = \sqrt{\frac{4.5092^3-2(4.5092)+5}{3}} = 5.4058..$
$x_3 = \sqrt{\frac{1.2544..^3-2(1.2544..)+5}{3}} = 1.2200..$	$x_3 = \sqrt{\frac{5.4058..^3-2(5.4058..)+5}{3}} = 7.1219..$
As you can see, when we used $x_0 = 1.5$, our iterations slowly converged. With $x_0 = 4$ however, each successive iteration moved further away, indicating divergence. This shows the effect that an unsuitable starting value can have.	

There are two ways in which an iteration can converge:

Way 1: Successive iterations get closer to the root, approaching the root from the same direction. This is graphically represented by a "staircase" diagram.

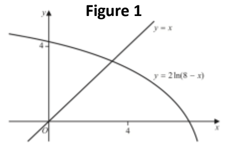


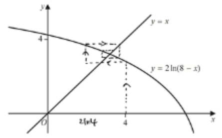
Way 2: Successive iterations alternate between being above and below the root. This is graphically represented by a "cobweb" diagram.



<u>Example 3:</u> Figure 1 shows the graph of $y = 2\ln(8 - x)$ and the graph of $y = x$. These curves meet at a single point, $x = \alpha$. A student uses the iteration formula $x_{n+1} = 2\ln(8 - x_n)$ in an attempt to find an approximation for α. Using the graph and starting with $x_1 = 4$, determine whether or not this iteration formula can be used to find an approximation for α.

Figure 1



We use the formula to find the first few iterations.	$x_1 = 2\ln(8-4) = 2ln4 = 2.773$ ... $x_2 = 2\ln(8-2ln4) = 3.308$ ... $x_3 = 2\ln(8-2ln4) = 3.092$ ...
Now we represent these iterations on the diagram: We can see that a cobweb diagram is formed. Each iteration gets closer and closer to the root which indicates convergence.	
Stating why our diagram means the iteration formula is suitable.	Since successive iterations form a cobweb diagram, the formula does indeed converge to the root α, so it is suitable.

Newton-Raphson method

Another method to find the roots of an equation $f(x) = 0$ is to use the Newton-Raphson method. The Newton-Raphson formula is:

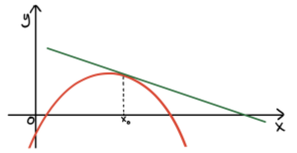
- $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

This method uses tangent lines to find accurate approximations of roots. The starting values must be chosen carefully with the Newton-Raphson method. Usually this will be given to you, but if not then you need to consider the following two points:

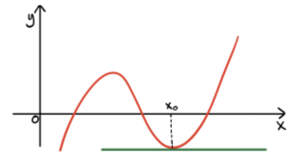
- If the starting value, x_0, is near a turning point then the method can converge on a root quite slowly, as the tangent line will be far from the x-axis.
- If the starting value, x_0, is at a turning point then the method will fail completely as the formula would result in division by 0, which is undefined.

The two graphs below illustrate each of the above cases:

Slow convergence
the starting point is near a turning point, so the tangent line cuts the x axis far from the root.



Failure
the starting value is at a turning point, so the tangent never cuts the x-axis.



<u>Example 4:</u> $f(x) = 2secx + 2x - 3$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ where x is in radians.
Given that $f(x) = 0$ has a solution, α, in the interval $0.4 < x < 0.5$, take 0.4 as a first approximation to α and apply the Newton-Raphson procedure to obtain a second approximation. Give your answer to 3 decimal places.

We start by finding $f'(x)$, then substituting $x = 0.4$ into $f'(x)$ and $f(x)$.	$f'(x) = 2secxtanx + 2 \Rightarrow f'(0.4) = 2\sec(0.4)\tan(0.4) + 2 = 2.9181..$ $f(0.4) = 2\sec(0.4) + 2(0.4) - 3 = -0.02859..$
Now using the Newton-Raphson formula:	$x_2 = 0.4 - \frac{-0.02859..}{2.9181..} = 0.4098.. = 0.410$ to 3 decimal places.

Applications to modelling

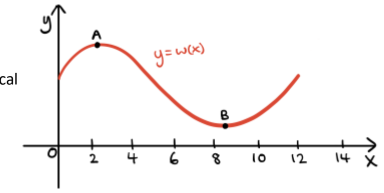
You also need to be able to apply your knowledge of numerical methods to questions involving models of real-life scenarios.

<u>Example 5:</u> The future world ranking position of a tennis player during a calendar year can be modelled by the function.

$$w(x) = -\frac{1}{50}x^4 + \frac{7}{10}x^3 - 7x^2 + 17x + 40, \quad 0 \le x \le 12$$

where x is the number of months since the beginning of the year.

The diagram shows the graph with equation $y = w(x)$. The graph has a local maximum at A and local minimum at B.



a) Find $w'(x)$.
b) Show that the player reaches a minimum ranking between 8.3 and 8.4 months after the beginning of the year.
c) Show that the turning points of the graph correspond to the equation $x = \pm\sqrt{\frac{10}{21}\left(\frac{2}{25}x^3 + 14x - 17\right)}$.

a) We use the formula to find the first few iterations.	$w'(x) = -\frac{4}{50}x^3 + \frac{21}{10}x^2 - 14x + 17$
b) We want to show there is a turning point between $x = 8.3$ and $x = 8.4$. This means we want to show $w'(x) = 0$ has a root α in the interval (8.3, 8.4). Using the technique from example 1:	$w'(8.3) = -\frac{4}{50}(8.3)^3 + \frac{21}{10}(8.3)^2 - 14(8.3) + 17 = -0.27396$ $w'(8.4) = -\frac{4}{50}(8.4)^3 + \frac{21}{10}(8.4)^2 - 14(8.4) + 17 = 0.15968$ There is a change in sign between $x = 8.3$ and $x = 8.4$ so a root must lie in this interval, which means that the player reaches a minimum ranking between 8.3 and 8.4 months. We know the point is a minimum since the graph only has a minimum in this range.
c)	$w'(x) = -\frac{4}{50}x^3 + \frac{21}{10}x^2 - 14x + 17 = 0$ $\frac{21}{10}x^2 = \frac{2}{25}x^3 + 14x - 17$ $x^2 = \frac{10}{21}\left(\frac{2}{25}x^3 + 14x - 17\right)$ $x = \pm\sqrt{\frac{10}{21}\left(\frac{2}{25}x^3 + 14x - 17\right)}$