6. Exponentials and Logarithms

Topics	What students need to learn:		
		Content	Guidance
6 **Exponentials and logarithms**	6.1	**Know and use the function a^x and its graph, where a is positive.** **Know and use the function e^x and its graph.**	**Understand the difference in shape between $a < 1$ and $a > 1$** To include the graph of $y = e^{ax+b} + c$
	6.2	**Know that the gradient of e^{kx} is equal to ke^{kx} and hence understand why the exponential model is suitable in many applications.**	**Realise that when the rate of change is proportional to the y value, an exponential model should be used.**
	6.3	**Know and use the definition of $\log_a x$ as the inverse of a^x, where a is positive and $x > 0$.** **Know and use the function $\ln x$ and its graph.** **Know and use $\ln x$ as the inverse function of e^x**	$a \neq 1$ **Solution of equations of the form $e^{ax+b} = p$ and $\ln(ax + b) = q$ is expected.**
	6.4	**Understand and use the laws of logarithms:** $\log_a x + \log_a y = \log_a (xy)$ $\log_a x - \log_a y = \log_a\left(\frac{x}{y}\right)$ $k \le \log_a x = \log_a x^k$ **(including, for example, $k = -1$ and $k = -\frac{1}{2}$)**	**Includes $\log_a a = 1$**
	6.5	**Solve equations of the form $a^x = b$**	**Students may use the change of base formula. Questions may be of the form, e.g. $2^{3x-1} = 3$**
	6.6	**Use logarithmic graphs to estimate parameters in relationships of the form** **$y = ax^n$ and $y = kb^x$, given data for x and y**	**Plot $\log y$ against $\log x$ and obtain a straight line where the intercept is $\log a$ and the gradient is n** **Plot $\log y$ against x and obtain a straight line where the intercept is $\log k$ and the gradient is $\log b$**
6 **Exponentials and logarithms** *continued*	6.7	**Understand and use exponential growth and decay; use in modelling (examples may include the use of e in continuous compound interest, radioactive decay, drug concentration decay, exponential growth as a model for population growth); consideration of limitations and refinements of exponential models.**	**Students may be asked to find the constants used in a model.** **They need to be familiar with terms such as initial, meaning when $t = 0$.** **They may need to explore the behaviour for large values of t or to consider whether the range of values predicted is appropriate.** **Consideration of a second improved model may be required.**

6. Exponentials and Logarithms

P 1 The points P and Q lie on the curve with equation $y = e^{\frac{1}{2}x}$.
The x-coordinates of P and Q are $\ln 4$ and $\ln 16$ respectively.

a Find an equation for the line PQ.

b Show that this line passes through the origin O.

c Calculate the length, to 3 significant figures, of the line segment PQ.

P 2 'he total number of views (in millions) V of a viral video in x days is modelled by

$$V = e^{0.4x} - 1$$

a Find the total number of views after 5 days, giving your answer to 2 significant figures.

b Find $\frac{dV}{dx}$.

P 3 The moment magnitude scale is used by seismologists to express the sizes of earthquakes.
The scale is calculated using the formula

$$M = \frac{2}{3}\log_{10}(S) - 10.7$$

where S is the seismic moment in dyne cm.

a Find the magnitude of an earthquake with a seismic moment of 2.24×10^{22} dyne cm.

b Find the seismic moment of an earthquake with
i magnitude 6 **ii** magnitude 7

c Using your answers to part **b** or otherwise, show that an earthquake of magnitude 7 is approximately 32 times as powerful as an earthquake of magnitude 6.

E/P 4 A student is asked to solve the equation

$$\log_2 x - \frac{1}{2}\log_2(x + 1) = 1$$

The student's attempt is shown

$$\begin{aligned}
\log_2 x - \log_2\sqrt{x+1} &= 1\\
x - \sqrt{x+1} &= 2^1\\
x - 2 &= \sqrt{x+1}\\
(x-2)^2 &= x + 1\\
x^2 - 5x + 3 &= 0\\
x = \frac{5+\sqrt{13}}{2} \quad & x = \frac{5-\sqrt{13}}{2}
\end{aligned}$$

a Identify the error made by the student. **(1 mark)**

b Solve the equation correctly. **(3 marks)**

6. Exponentials and Logarithms

5 The population, P, of a colony of endangered Caledonian owlet-nightjars can be modelled by the equation $P = ab^t$ where a and b are constants and t is the time, in months, since the population was first recorded.

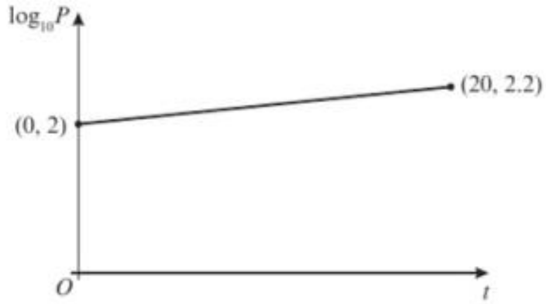


Figure 2

The line l shown in figure 2 shows the relationship between t and $\log_{10}P$ for the population over a period of 20 years.

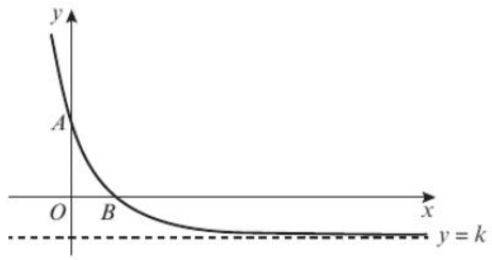
a Write down an equation of line l. **(3)**

b Work out the value of a and interpret this value in the context of the model. **(3)**

c Work out the value of b, giving your answer correct to 3 decimal places. **(2)**

d Find the population predicted by the model when $t = 30$. **(1)**

E 6 The graph of the function $f(x) = 3e^{-x} - 1$, $x \in \mathbb{R}$, has an asymptote $y = k$, and crosses the x and y axes at A and B respectively, as shown in the diagram.



a Write down the value of k and the y-coordinate of A. **(2 marks)**

b Find the exact value of the x-coordinate of B, giving your answer as simply as possible. **(2 marks)**

6. Exponentials and Logarithms

Answers

1 **a** $y = \left(\frac{2}{\ln 4}\right)x$

b (0, 0) satisfies the equation of the line.

c 2.43

2 **a** 6.4 million views

b $\frac{dV}{dx} = 0.4e^{0.4x}$

c 9.42×10^{16} new views per day

d This is too big, so the model is not valid after 100 days

3 **a** 4.2

b **i** 1.12×10^{25} dyne cm

ii 3.55×10^{26} dyne cm

c divide **b ii** by **b i**

4 **a** They exponentiated the two terms on LHS separately rather than combining them first.

b $x = 2 + 2\sqrt{2}$

5 **a** $\log_{10} P = 0.01t + 2$

b 100, initial population

c 1.023

d Accept answers from 195 to 200

6 **a** $k = -1$, $A(0, 2)$

b $\ln 3$

7. Differentiation

Topics	What students need to learn: Content		Guidance
7 Differentiation	**7.1**	**Understand and use the derivative of $f(x)$ as the gradient of the tangent to the graph of $y = f(x)$ at a general point (x, y); the gradient of the tangent as a limit; interpretation as a rate of change**	**Know that $\frac{dy}{dx}$ is the rate of change of y with respect to x.** **The notation $f'(x)$ may be used for the first derivative and $f''(x)$ may be used for the second derivative.**
		sketching the gradient function for a given curve	**Given for example the graph of $y = f(x)$, sketch the graph of $y = f'(x)$ using given axes and scale. This could relate speed and acceleration for example.**
		second derivatives	
		differentiation from first principles for small positive integer powers of x and for $\sin x$ and $\cos x$	**For example, students should be able to use, for $n = 2$ and $n = 3$, the gradient expression** $\lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$ **Students may use δx or h**
7 Differentiation *continued*	7.1 *cont.*	**Understand and use the second derivative as the rate of change of gradient;** connection to convex and concave sections of curves and points of inflection.	Use the condition $f''(x) > 0$ implies a minimum and $f''(x) < 0$ implies a maximum for points where $f'(x) = 0$ Know that at an inflection point $f''(x)$ changes sign. Consider cases where $f''(x) = 0$ and $f'(x) = 0$ where the point may be a minimum, a maximum or a point of inflection (e.g. $y = x^n, n > 2$)
	7.2	**Differentiate x^n, for rational values of n, and related constant multiples, sums and differences.**	**For example, the ability to differentiate expressions such as** $(2x+5)(x-1)$ **and** $\frac{x^2+3x-5}{4x^{\frac{1}{2}}}, x > 0,$ **is expected.**
		Differentiate e^{kx} and a^{kx}, $\sin kx$, $\cos kx$, $\tan kx$ and related sums, differences and constant multiples.	Knowledge and use of the result $\frac{d}{dx}(a^{kx}) = ka^{kx}\ln a$ is expected.
		Understand and use the derivative of $\ln x$	
	7.3	**Apply differentiation to find gradients, tangents and normals**	**Use of differentiation to find equations of tangents and normals at specific points on a curve.**
		maxima and minima and stationary points. points on inflection	**To include applications to curve sketching. Maxima and minima problems may be set in the context of a practical problem.**
		Identify where functions are increasing or decreasing.	**To include applications to curve sketching.**
	7.4	Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions.	Differentiation of $\operatorname{cosec} x$, $\cot x$ and $\sec x$ and differentiation of $\arcsin x$, $\arccos x$, and $\arctan x$ are required. Skill will be expected in the differentiation of functions generated from standard forms using products, quotients and composition, such as $2x^4 \sin x$, $\frac{e^{3x}}{x}$, $\cos^2 x$ and $\tan^2 2x$.

7. Differentiation

1 A curve has equation $y = \frac{8}{x} - x + 3x^2$, $x > 0$. Find the equations of the tangent and the normal to the curve at the point where $x = 2$.

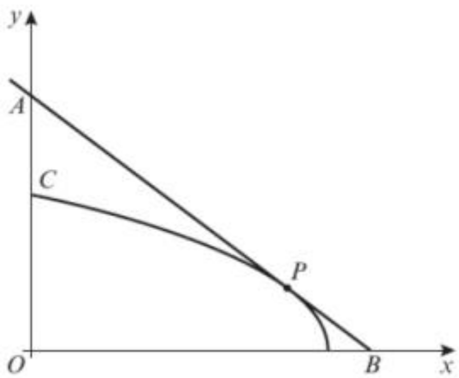
P 2 The total surface area, A cm^2, of a cylinder with a fixed volume of 1000 cm^3 is given by the formula $A = 2\pi x^2 + \frac{2000}{x}$, where x cm is the radius. Show that when the rate of change of the area with respect to the radius is zero, $x^3 = \frac{500}{\pi}$

E/P 3 Given that a curve has equation $y = \cos^2 x + \sin x$, $0 < x < 2\pi$, find the coordinates of the stationary points of the curve. **(6 marks)**

E/P 4 The diagram shows the curve C with parametric equations

$$x = a\sin^2 t, \quad y = a\cos t, \quad 0 \leqslant t \leqslant \tfrac{1}{2}\pi$$

where a is a positive constant. The point P lies on C and has coordinates $\left(\frac{3}{4}a, \frac{1}{2}a\right)$.



a Find $\frac{dy}{dx}$, giving your answer in terms of t. **(4 marks)**

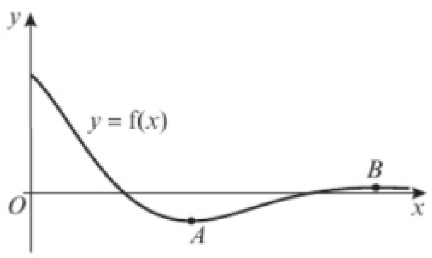
b Find an equation of the tangent to C at P. **(4 marks)**

The tangent to C at P cuts the coordinate axes at points A and B.

c Show that the triangle AOB has area ka^2 where k is a constant to be found. **(2 marks)**

E/P 5 A curve has equation $7x^2 + 48xy - 7y^2 + 75 = 0$. A and B are two distinct points on the curve and at each of these points the gradient of the curve is equal to $\frac{2}{11}$. Use implicit differentiation to show that the straight line passing through A and B has equation $x + 2y = 0$. **(6 marks)**

E/P 6 The curve C with equation $y = f(x)$ is shown in the diagram, where $f(x) = \frac{\cos 2x}{e^x}$, $0 \leqslant x \leqslant \pi$



The curve has a local minimum at A and a local maximum at B.

a Show that the x-coordinates of A and B satisfy the equation $\tan 2x = -0.5$ and hence find the coordinates of A and B. **(6 marks)**

b Using your answer to part **a**, find the coordinates of the maximum and minimum turning points on the curve with equation $y = 2 + 4f(x - 4)$. **(3 marks)**

c Determine the values of x for which $f(x)$ is concave. **(5 marks)**

7. Differentiation

Answers

1 $y = 9x - 4$ and $9y + x = 128$

3 $\left(\frac{\pi}{6}, \frac{5}{4}\right), \left(\frac{\pi}{2}, 1\right), \left(\frac{5\pi}{6}, \frac{5}{4}\right), \left(\frac{3\pi}{2}, -1\right)$

4 **a** $-\frac{1}{2}\sec t$ **b** $4y + 4x = 5a$

c Tangent crosses the x-axis at $x = \frac{5}{4}a$, and crosses the y-axis at $y = \frac{5}{4}a$.

So area $AOB = \frac{1}{2}\left(\frac{5}{4}a\right)^2 = \frac{25}{32}a^2$, $k = \frac{25}{32}$

5 $14x + 48y + 48x\frac{dy}{dx} - 14y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-7x - 24y}{24x - 7y}$

So $\frac{-7x - 24y}{24x - 7y} = \frac{2}{11} \Rightarrow -77x - 264y$

$= 48x - 14y \Rightarrow x + 2y = 0$

6 **a** $f'(x) = -\frac{2\sin 2x + \cos 2x}{e^x}$

$f'(x) = 0 \Leftrightarrow 2\sin 2x + \cos 2x = 0 \Leftrightarrow \tan 2x = -0.5$

$A(1.34, -0.234)$, $B(2.91, 0.0487)$

b Maximum $(6.91, 2.19)$; minimum $(5.34, 1.06)$ to 3 s.f.

c $0 < x \leqslant 0.322$, $1.89 \leqslant x < \pi$

8. Integration

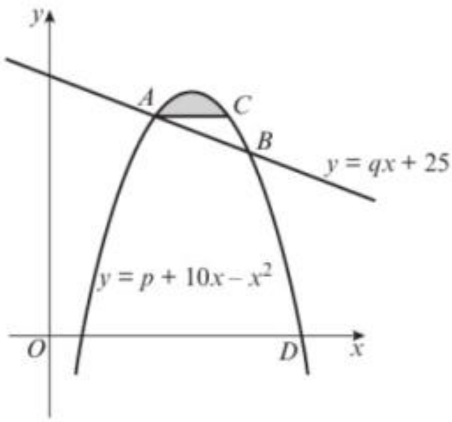
Topics	What students need to learn:		
		Content	Guidance
8 **Integration**	8.1	**Know and use the Fundamental Theorem of Calculus**	**Integration as the reverse process of differentiation. Students should know that for indefinite integrals a constant of integration is required.**
	8.2	**Integrate x^n (excluding $n = -1$) and related sums, differences and constant multiples.**	**For example, the ability to integrate expressions such as $\frac{1}{2}x^2 - 3x^{-\frac{1}{2}}$ and $\frac{(x+2)^2}{x^{\frac{1}{2}}}$ is expected.** **Given $f'(x)$ and a point on the curve, Students should be able to find an equation of the curve in the form $y = f(x)$.**
		Integrate e^{kx}, $\frac{1}{x}$, $\sin kx$, $\cos kx$ and related sums, differences and constant multiples.	To include integration of standard functions such as $\sin 3x$, $\sec^2 2x$, $\tan x$, e^{5x}, $\frac{1}{2x}$. Students are expected to be able to use trigonometric identities to integrate, for example, $\sin^2 x$, $\tan^2 x$, $\cos^2 3x$.
	8.3	**Evaluate definite integrals; use a definite integral to find the area under a curve** and the area between two curves	Students will be expected to be able to evaluate the area of a region bounded by a curve and given straight lines, or between two curves. This includes curves defined parametrically. For example, find the finite area bounded by the curve $y = 6x - x^2$ and the line $y = 2x$ Or find the finite area bounded by the curve $y = x^2 - 5x + 6$ and the curve $y = 4 - x^2$.
	8.4	Understand and use integration as the limit of a sum.	Recognise $\int_a^b f(x)\,dx = \lim_{\delta x \to 0} \sum_{x=a}^{b} f(x)\,\delta x$

8. Integration

Topics	What students need to learn:		
	Content		Guidance
8 **Integration** *continued*	8.5	Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.)	Students should recognise integrals of the form $\int \frac{f'(x)}{f(x)}dx = \ln f(x) + c$. The integral $\int \ln x \, dx$ is required Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae.
	8.6	Integrate using partial fractions that are linear in the denominator.	Integration of rational expressions such as those arising from partial fractions, e.g. $\frac{2}{3x+5}$ Note that the integration of other rational expressions, such as $\frac{x}{x^2+5}$ and $\frac{2}{(2x-1)^4}$ is also required (see previous paragraph).
	8.7	Evaluate the analytical solution of simple first order differential equations with separable variables, including finding particular solutions (Separation of variables may require factorisation involving a common factor.)	Students may be asked to sketch members of the family of solution curves.
	8.8	Interpret the solution of a differential equation in the context of solving a problem, including identifying limitations of the solution; includes links to kinematics.	The validity of the solution for large values should be considered.

8. Integration

(E/P) 1 The diagram shows part of the curve with equation $y = p + 10x - x^2$, where p is a constant, and part of the line l with equation $y = qx + 25$, where q is a constant. The line l cuts the curve at the points A and B. The x-coordinates of A and B are 4 and 8 respectively. The line through A parallel to the x-axis intersects the curve again at the point C.



a Show that $p = -7$ and calculate the value of q. **(3 marks)**

b Calculate the coordinates of C. **(2 marks)**

c The shaded region in the diagram is bounded by the curve and the line segment AC. Using integration and showing all your working, calculate the area of the shaded region. **(6 marks)**

(E) 2 Using the substitution $t^2 = x + 1$, where $x > -1$,

a find $\int \frac{x}{\sqrt{x+1}}\,dx$. **(5 marks)**

b Hence evaluate $\int_0^3 \frac{x}{\sqrt{x+1}}\,dx$. **(2 marks)**

(E) 3 **a** Use integration by parts to find $\int x \sin 8x\,dx$. **(4 marks)**

b Use your answer to part **a** to find $\int x^2 \cos 8x\,dx$. **(4 marks)**

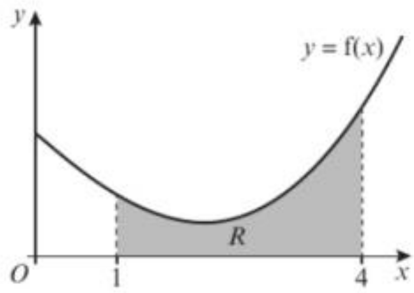
(E/P) 4 $f(x) = \frac{5x^2 - 8x + 1}{2x(x-1)^2}$

a Given that $f(x) = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$, find the values of the constants A, B and C. **(4 marks)**

b Hence find $\int f(x)\,dx$. **(4 marks)**

c Hence show that $\int_4^9 f(x)\,dx = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$ **(4 marks)**

E/P 5 The diagram shows a sketch of the curve $y = f(x)$, where $f(x) = \frac{1}{5}x^2 \ln x - x + 2, x > 0$.
The region R, shown in the diagram, is bounded by the curve, the x-axis and the lines with equations $x = 1$ and $x = 4$.



The table below shows the corresponding values of x and y with the y values given to 4 decimal places as appropriate.

x	1	1.5	2	2.5	3	3.5	4
y	1	0.6825	0.5545	0.6454		1.5693	2.4361

a Complete the table with the missing value of y. **(1 mark)**

b Use the trapezium rule, with all the values of y in the table, to obtain an estimate for the area of R, giving your answer to 3 decimal places. **(3 marks)**

c Explain how the trapezium rule could be used to obtain a more accurate estimate for the area of R. **(1 mark)**

d Show that the exact area of R can be written in the form $\frac{a}{b} + \frac{c}{d} \ln e$, where a, b, c, d and e are integers. **(6 marks)**

e Find the percentage error in the answer in part **b**. **(2 marks)**

E/P 6 An oil spill is modelled as a circular disc with radius r km and area A km². The rate of increase of the area of the oil spill, in km²/day at time t days after it occurs is modelled as:

$$\frac{dA}{dt} = k \sin\left(\frac{t}{3\pi}\right), 0 \leqslant t \leqslant 12$$

a Show that $\frac{dr}{dt} = \frac{k}{2\pi r} \sin\left(\frac{t}{3\pi}\right)$ **(2 marks)**

Given that the radius of the spill at time $t = 0$ is 1 km, and the radius of the spill at time $t = \pi^2$ is 2 km:

b find an expression for r^2 in terms of t **(7 marks)**

c find the time, in days and hours to the nearest hour, after which the radius of the spill is 1.5 km. **(3 marks)**

8. Integration

Answers

1 **a** $q = -2$ **b** $C(6,17)$ **c** $1\frac{1}{3}$

2 **a** $\frac{2}{3}(x-2)\sqrt{x+1}+c$ **b** $\frac{8}{3}$

3 **a** $-\frac{1}{8}x\cos 8x + \frac{1}{64}\sin 8x + c$

b $\frac{1}{8}x^2\sin 8x + \frac{1}{32}x\cos 8x - \frac{1}{256}\sin 8x + c$

4 **a** $A = \frac{1}{2}, B = 2, C = -1$

b $\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1} + c$

c $\int_4^9 f(x)\,dx = \left[\frac{1}{2}\ln|x| + 2\ln|x-1| + \frac{1}{x-1}\right]_4^9$

$$= \left(\frac{1}{2}\ln 9 + 2\ln 8 + \frac{1}{8}\right) - \left(\frac{1}{2}\ln 4 + 2\ln 3 + \frac{1}{3}\right)$$

$$= \left(\ln 3 + \ln 64 + \frac{1}{8}\right) - \left(\ln 2 + \ln 9 + \frac{1}{3}\right)$$

$$= \ln\left(\frac{3\times 64}{2\times 9}\right) - \frac{5}{24} = \ln\left(\frac{32}{3}\right) - \frac{5}{24}$$

5 **a** 0.9775 **b** 3.074

c Use more values, use smaller intervals. The lines would then more closely follow the curve.

d $\int_1^4 \left(\frac{1}{5}x^2\right)\ln x - x + 2\,dx$

$$= \left[\frac{1}{15}x^3\ln x - \frac{1}{45}x^3 - \frac{1}{2}x^2 + 2x\right]_1^4$$

$$= \left(\frac{64}{15}\ln 4 - \frac{64}{45}\right) - \left(-\frac{1}{45} - \frac{1}{2} + 2\right) = \frac{-29}{10} + \frac{64}{15}\ln 4$$

e 2.0%

6 **a** $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

$$\frac{dr}{dt} = \frac{dr}{dA}\times\frac{dA}{dt} = \frac{1}{2\pi r}\times k\sin\left(\frac{t}{3\pi}\right) = \frac{k}{2\pi r}\sin\left(\frac{t}{3\pi}\right)$$

b $r^2 = -6\cos\left(\frac{t}{3\pi}\right) + 7$ **c** 6 days, 5 hours

9. Numerical Methods

Topics	What students need to learn:		
	Content		Guidance
9 **Numerical methods**	9.1	Locate roots of $f(x) = 0$ by considering changes of sign of $f(x)$ in an interval of x on which $f(x)$ is sufficiently well behaved. Understand how change of sign methods can fail.	Students should know that sign change is appropriate for continuous functions in a small interval. When the interval is too large sign may not change as there may be an even number of roots. If the function is not continuous, sign may change but there may be an asymptote (not a root).
	9.2	Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams.	Understand that many mathematical problems cannot be solved analytically, but numerical methods permit solution to a required level of accuracy. Use an iteration of the form $x_{n+1} = f(x_n)$ to find a root of the equation $x = f(x)$ and show understanding of the convergence in geometrical terms by drawing cobweb and staircase diagrams.
	9.2	Solve equations using the Newton-Raphson method and other recurrence relations of the form $x_{n+1} = g(x_n)$ Understand how such methods can fail.	For the Newton-Raphson method, students should understand its working in geometrical terms, so that they understand its failure near to points where the gradient is small.
	9.3	Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between.	For example, evaluate $\int_0^1 \sqrt{(2x+1)}\ \mathrm{d}x$ using the values of $\sqrt{(2x+1)}$ at $x = 0$, 0.25, 0.5, 0.75 and 1 and use a sketch on a given graph to determine whether the trapezium rule gives an over-estimate or an under-estimate.
	9.4	Use numerical methods to solve problems in context.	Iterations may be suggested for the solution of equations not soluble by analytic means.

9. Numerical Methods

E/P 1 $f(x) = x^3 - 6x - 2$

a Show that the equation $f(x) = 0$ can be written in the form $x = \pm\sqrt{a + \frac{b}{x}}$, and state the values of the integers a and b. **(2 marks)**

$f(x) = 0$ has one positive root, α.

The iterative formula $x_{n+1} = \sqrt{a + \frac{b}{x_n}}$, $x_0 = 2$ is used to find an approximate value for α.

b Calculate the values of x_1, x_2, x_3 and x_4 to 4 decimal places. **(3 marks)**

c By choosing a suitable interval, show that $\alpha = 2.602$ is correct to 3 decimal places. **(3 marks)**

E/P 2 $g(x) = x^2 - 3x - 5$

a Show that the equation $g(x) = 0$ can be written as $x = \sqrt{3x + 5}$. **(1 mark)**

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{3x + 5}$. **(2 marks)**

c Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{3x_n + 5}$ converges to a root of $g(x)$ when $x_0 = 1$. **(1 mark)**

$g(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \dfrac{x_n^2 - 5}{3}$

d With reference to a diagram, explain why this iterative formula diverges when $x_0 = 7$. **(3 marks)**

E/P 3 $g(x) = x^3 - 7x^2 + 2x + 4$

a Find $g'(x)$. **(2 marks)**

A root α of the equation $g(x) = 0$ lies in the interval [6.5, 6.7].

b Taking 6.6 as a first approximation to α, apply the Newton–Raphson process once to $g(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places. **(4 marks)**

c Given that $g(1) = 0$, find the exact value of the other two roots of $g(x)$. **(3 marks)**

d Calculate the percentage error of your answer in part **b**. **(2 marks)**

E/P 4 $f(x) = e^{0.8x} - \dfrac{1}{3 - 2x}$, $x \neq \dfrac{3}{2}$

a Show that the equation $f(x) = 0$ can be written as $x = 1.5 - 0.5e^{-0.8x}$. **(3 marks)**

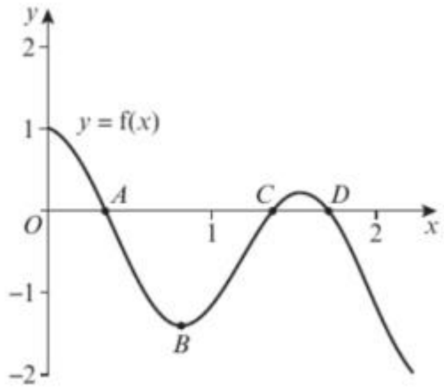
b Use the iterative formula $x_{n+1} = 1.5 - 0.5e^{-0.8x_n}$ with $x_0 = 1.3$ to obtain x_1, x_2 and x_3. Hence write down one root of $f(x) = 0$ correct to 3 decimal places. **(2 marks)**

c Show that the equation $f(x) = 0$ can be written in the form $x = p\ln(3 - 2x)$, stating the value of p. **(3 marks)**

d Use the iterative formula $x_{n+1} = p\ln(3 - 2x_n)$ with $x_0 = -2.6$ and the value of p found in part **c** to obtain x_1, x_2 and x_3. Hence write down a second root of $f(x) = 0$ correct to 2 decimal places. **(2 marks)**

9. Numerical Methods

E/P 5 The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = \cos(4x) - \frac{1}{2}x$.



a Show that the curve has a root in the interval [1.3, 1.4]. **(2 marks)**

b Use differentiation to find the coordinates of point B. Write each coordinate correct to 3 decimal places. **(3 marks)**

c Using the iterative formula $x_{n+1} = \frac{1}{4}\arccos\left(\frac{1}{2}x_n\right)$, with $x_0 = 0.4$, find the values of x_1, x_2, x_3 and x_4. Give your answers to 4 decimal places. **(3 marks)**

d Using $x_0 = 1.7$ as a first approximation to the root at D, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to the root, giving your answer to 3 decimal places. **(4 marks)**

e By considering the change of sign of $f(x)$ over an appropriate interval, show that the answer to part **d** is accurate to 3 decimal places. **(2 marks)**

E/P 6 **a** On the same axes, sketch the graphs of $y = \frac{1}{x}$ and $y = x + 3$. **(2 marks)**

b Write down the number of roots of the equation $\frac{1}{x} = x + 3$. **(1 mark)**

c Show that the positive root of the equation $\frac{1}{x} = x + 3$ lies in the interval (0.30, 0.31). **(2 marks)**

d Show that the equation $\frac{1}{x} = x + 3$ may be written in the form $x^2 + 3x - 1 = 0$. **(2 marks)**

e Use the quadratic formula to find the positive root of the equation $x^2 + 3x - 1 = 0$ to 3 decimal places. **(2 marks)**

9. Numerical Methods

Answers

1 **a** $x^3 - 6x - 2 = 0 \Rightarrow x^3 = 6x + 2$

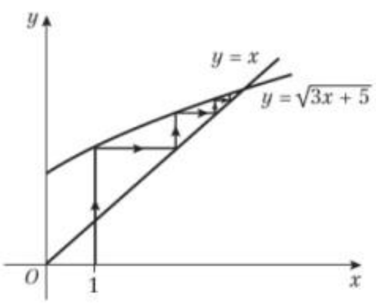
$\Rightarrow x^2 = 6 + \dfrac{2}{x} \Rightarrow x = \pm\sqrt{6 + \dfrac{2}{x}}$; $a = 6$, $b = 2$

b $x_1 = 2.6458, x_2 = 2.5992, x_3 = 2.6018, x_4 = 2.6017$

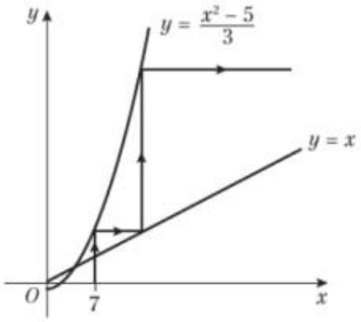
c $f(2.6015) = (2.6015)^3 - 6(2.6015) - 2 = -0.0025... < 0$
$f(2.6025) = (2.6025)^3 - 6(2.6025) - 2 = 0.0117 > 0$
There is a sign change in the interval $2.6015 < x < 2.6025$, so this implies there is a root in the interval.

2 **a** $g(x) = 0 \Rightarrow x^2 - 3x - 5 = 0$
$\Rightarrow x^2 = 3x + 5 \Rightarrow x = \sqrt{3x + 5}$

b, c



d



9. Numerical Methods

3 **a** $g'(x) = 3x^2 - 14x + 2$ **b** 6.606

c $(x - 1)(x^2 - 6x - 4) \Rightarrow x^2 - 6x - 4 = 0 \Rightarrow x = 3 \pm \sqrt{13}$

d 0.007%

4 **a** $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow (3 - 2x)e^{0.8x} - 1 = 0$

$\Rightarrow (3 - 2x)e^{0.8x} = 1 \Rightarrow 3 - 2x = e^{-0.8x}$

$\Rightarrow 3 - e^{-0.8x} = 2x \Rightarrow x = 1.5 - 0.5e^{-0.8x}$

b $x_1 = 1.32327\ldots$, $x_2 = 1.32653\ldots$, $x_3 = 1.32698\ldots$, root $= 1.327$ (3 d.p.)

c $e^{0.8x} - \frac{1}{3 - 2x} = 0 \Rightarrow e^{0.8x} = \frac{1}{3 - 2x} \Rightarrow 3 - 2x = e^{-0.8x}$

$\Rightarrow -0.8x = \ln(3 - 2x) \Rightarrow x = -1.25\ln(3 - 2x)$

$p = -1.25$

d $x_1 = -2.6302$, $x_2 = -2.6393$, $x_3 = -2.6421$, root $= -2.64$ (2 d.p.)

5 **a** $f(1.3) = -0.18148\ldots$, $f(1.4) = 0.07556\ldots$.. There is a sign change in the interval [1.3, 1.4], so there is a root in this interval.

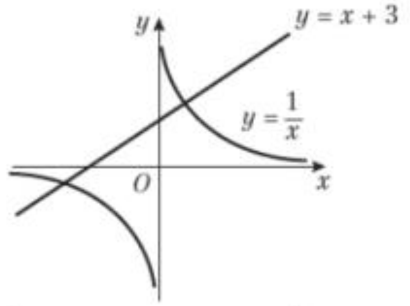
b (0.817, −1.401)

c $x_1 = 0.3424$, $x_2 = 0.3497$, $x_3 = 0.3488$, $x_4 = 0.3489$

d $x_1 = 1.708$

e $f(1.7075) = 0.000435\ldots$., $f(1.7085) = -0.002151\ldots$. There is a sign change in the interval [1.7075, 1.7085], so there is a root in this interval.

6 **a**



b 2

c $\frac{1}{x} = x + 3 \Rightarrow 0 = x + 3 - \frac{1}{x}$, let $f(x) = x + 3 - \frac{1}{x}$

$f(0.30) = -0.0333\ldots < 0$, $f(0.31) = 0.0841\ldots > 0$.

Sign change implies root.

d $\frac{1}{x} = x + 3 \Rightarrow 1 = x^2 + 3x \Rightarrow 0 = x^2 + 3x - 1$

e 0.303

10. Vectors

Topics	What students need to learn:		
	Content		Guidance
10 **Vectors**	10.1	**Use vectors in two dimensions** and in three dimensions	**Students should be familiar with column vectors and with the use of i and j unit vectors in two dimensions** and **i**, **j** and **k** unit vectors in three dimensions.
	10.2	**Calculate the magnitude and direction of a vector and convert between component form and magnitude/direction form.**	**Students should be able to find a unit vector in the direction of a, and be familiar with the notation $\lvert a \rvert$.**
	10.3	**Add vectors diagrammatically and perform the algebraic operations of vector addition and multiplication by scalars, and understand their geometrical interpretations.**	**The triangle and parallelogram laws of addition.** **Parallel vectors.**
	10.4	**Understand and use position vectors; calculate the distance between two points represented by position vectors.**	$\overrightarrow{OB} - \overrightarrow{OA} = \overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ **The distance d between two points (x_1, y_1) and (x_2, y_2) is given by** $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$
10 **Vectors**	10.5	**Use vectors to solve problems in pure mathematics and in context, (including forces).**	**For example, finding position vector of the fourth corner of a shape (e.g. parallelogram) $ABCD$ with three given position vectors for the corners A, B and C.** **Or use of ratio theorem to find position vector of a point C dividing AB in a given ratio.** **Contexts such as velocity, displacement, kinematics and forces will be covered in Paper 3, Sections 6.1, 7.3 and 8.1 – 8.4**

10. Vectors

E/P 1 The resultant of the vectors $\mathbf{a} = 4\mathbf{i} - 3\mathbf{j}$ and $\mathbf{b} = 2p\mathbf{i} - p\mathbf{j}$ is parallel to the vector $\mathbf{c} = 2\mathbf{i} - 3\mathbf{j}$. Find:

a the value of p **(3 marks)**

b the resultant of vectors **a** and **b**. **(1 mark)**

E/P 2 Two forces, $\mathbf{F}_1$ and $\mathbf{F}_2$, are given by the vectors $\mathbf{F}_1 = (4\mathbf{i} - 5\mathbf{j})$ N and $\mathbf{F}_2 = (p\mathbf{i} + q\mathbf{j})$ N. The resultant force, $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2$ acts in a direction which is parallel to the vector $(3\mathbf{i} - \mathbf{j})$

a Find the angle between **R** and the vector **i**. **(3 marks)**

b Show that $p + 3q = 11$. **(4 marks)**

c Given that $p = 2$, find the magnitude of **R**. **(2 marks)**

E 3 P is the point $(-6, 2, 1)$, Q is the point $(3, -2, 1)$ and R is the point $(1, 3, -2)$.

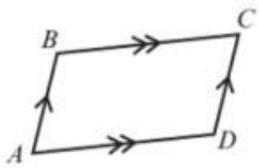
a Find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{QR}$. **(3 marks)**

b Hence find the lengths of the sides of triangle PQR. **(6 marks)**

c Given that angle $QRP = 90°$ find the size of angle PQR. **(2 marks)**

E/P 4 The diagram shows the quadrilateral $ABCD$.

Given that $\overrightarrow{AB} = \begin{pmatrix} 6 \\ -2 \\ 11 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 15 \\ 8 \\ 5 \end{pmatrix}$, find the area of the quadrilateral. **(7 marks)**



P 5 A is the point $(2, 3, -2)$, B is the point $(0, -2, 1)$ and C is the point $(4, -2, -5)$. When A is reflected in the line BC it is mapped to the point D.

a Work out the coordinates of the point D.

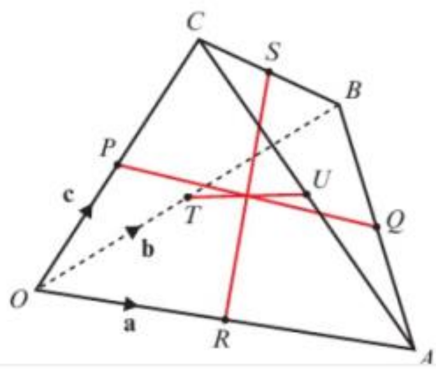
b Give the mathematical name for the shape $ABCD$.

c Work out the area of $ABCD$.

P 6 The diagram shows a tetrahedron $OABC$. **a**, **b** and **c** are the position vectors of A, B and C respectively.

P, Q, R, S, T and U are the midpoints of OC, AB, OA, BC, OB and AC respectively.

Prove that the line segments PQ, RS and TU meet at a point and bisect each other.



10. Vectors

Answers

1 **a** $p = -1.5$ **b** $\mathbf{i} - 1.5\mathbf{j}$

2 **a** 18.4° below

b $\mathbf{R} = (4 + p)\mathbf{i} + (-5 + q)\mathbf{j}$, $4 + p = 3\lambda$ and $-5 + q = -\lambda$

$4 + p = 3(q - 5)$ so $p + 3q = 11$

c $2\sqrt{10} = 6.32$ newtons

3 **a** $\overrightarrow{PQ} = 9\mathbf{i} - 4\mathbf{j}$, $\overrightarrow{PR} = 7\mathbf{i} + \mathbf{j} - 3\mathbf{k}$, $\overrightarrow{QR} = -2\mathbf{i} + 5\mathbf{j} - 3\mathbf{k}$

b $|\overrightarrow{PQ}| = \sqrt{97}$, $|\overrightarrow{PR}| = \sqrt{59}$, $|\overrightarrow{QR}| = \sqrt{38}$ **c** 51.3°

4

184 (3 s.f.)

5 **a** (2, −7, −2) **b** rhombus **c** 36.1

6 $\overrightarrow{PQ} = \frac{1}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$, $\overrightarrow{RS} = \frac{1}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$, $\overrightarrow{TU} = \frac{1}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Let $\overrightarrow{PQ}$, $\overrightarrow{RS}$ and $\overrightarrow{TU}$ intersect at X: $\overrightarrow{PX} = r\overrightarrow{PQ} = \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\overrightarrow{RX} = s\overrightarrow{RS} = \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c})$

$\overrightarrow{TX} = t\overrightarrow{TU} = \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c})$ for scalars r, s and t

$\overrightarrow{RX} = \overrightarrow{RO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{s}{2}(-\mathbf{a} + \mathbf{b} + \mathbf{c}) = \frac{1}{2}(-\mathbf{a} + \mathbf{c}) + \frac{r}{2}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $r = s = \frac{1}{2}$

$\overrightarrow{TX} = \overrightarrow{TO} + \overrightarrow{OP} + \overrightarrow{PX} = \frac{1}{2}(-\mathbf{b} + \mathbf{c}) + \frac{1}{4}(\mathbf{a} + \mathbf{b} - \mathbf{c})$

$\Rightarrow \frac{t}{2}(\mathbf{a} - \mathbf{b} + \mathbf{c}) = \frac{1}{4}(\mathbf{a} - \mathbf{b} + \mathbf{c})$

Comparing coefficients in **a**, **b** and **c** gives $t = \frac{1}{2}$

So the line segments PQ, RS and TU meet at a point and bisect each other.