## Exercise 8A

For all questions in this exercise, take $\mathbf{i}$ and $\mathbf{j}$ to be the unit vectors due east and north respectively.

**1** A particle $P$ starts at the point with position vector $\mathbf{r}_0$. $P$ moves with constant velocity $\mathbf{v}$ m s$^{-1}$. After $t$ seconds, $P$ is at the point with position vector $\mathbf{r}$.

**a** Find $\mathbf{r}$ if $\mathbf{r}_0 = 2\mathbf{i}$, $\mathbf{v} = \mathbf{i} + 3\mathbf{j}$, and $t = 4$.

**b** Find $\mathbf{r}$ if $\mathbf{r}_0 = 3\mathbf{i} - \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j}$, and $t = 5$.

**c** Find $\mathbf{r}_0$ if $\mathbf{r} = 4\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$, and $t = 3$.

**d** Find $\mathbf{r}_0$ if $\mathbf{r} = -2\mathbf{i} + 5\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 3\mathbf{j}$, and $t = 6$.

**e** Find $\mathbf{v}$ if $\mathbf{r}_0 = 2\mathbf{i} + 2\mathbf{j}$, $\mathbf{r} = 8\mathbf{i} - 7\mathbf{j}$, and $t = 3$.

**f** Find $t$ if $\mathbf{r}_0 = 4\mathbf{i} + \mathbf{j}$, $\mathbf{r} = 12\mathbf{i} - 11\mathbf{j}$, and $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$.

**2** A radio-controlled boat starts from position vector $(10\mathbf{i} - 5\mathbf{j})$ m relative to a fixed origin and travels with constant velocity, passing a point with position vector $(-2\mathbf{i} + 9\mathbf{j})$ m after 4 seconds. Find the speed and bearing of the boat.

**3** A clockwork mouse starts from a point with position vector $(-2\mathbf{i} + 3\mathbf{j})$ m relative to a fixed origin and moves in a straight line with a constant speed of $4\,\text{m}\,\text{s}^{-1}$. Find the time taken for the mouse to travel to the point with position vector $(6\mathbf{i} - 3\mathbf{j})$ m.

**4** A helicopter starts from the point with position vector $\begin{pmatrix} 120 \\ -10 \end{pmatrix}$ m relative to a fixed origin, and moves with constant velocity $\begin{pmatrix} -30 \\ 40 \end{pmatrix}$ $\text{m}\,\text{s}^{-1}$. Find:

**a** the position vector of the helicopter $t$ seconds later

**b** the time at which the helicopter is due north of the origin.

**Hint** When the helicopter is due north of the origin, the $\mathbf{i}$-component of its position vector will be 0.

**5** At time $t = 0$, the particle $P$ is at the point with position vector $4\mathbf{i}$, and moving with constant velocity $\mathbf{i} + \mathbf{j}\,\text{m}\,\text{s}^{-1}$. A second particle $Q$ is at the point with position vector $-3\mathbf{j}$ and moving with velocity $\mathbf{v}\,\text{m}\,\text{s}^{-1}$. After 8 seconds, the paths of $P$ and $Q$ meet. Find the speed of $Q$.

**6** At noon, a ferry $F$ is 400 m due north of an observation point $O$ and is moving with a constant velocity of $(7\mathbf{i} + 7\mathbf{j})\,\text{m}\,\text{s}^{-1}$, and a speedboat $S$ is 500 m due east of $O$, moving with a constant velocity of $(-3\mathbf{i} + 15\mathbf{j})\,\text{m}\,\text{s}^{-1}$.

## Exercise 8B

For all questions in this exercise **i** and **j** are unit vectors horizontally and vertically respectively.

Unless stated otherwise, take $g = 9.8\,\text{m}\,\text{s}^{-2}$.

**1** A particle $P$ is projected from the origin with velocity $(12\mathbf{i} + 24\mathbf{j})\,\text{m}\,\text{s}^{-1}$. The particle moves freely under gravity. Find:

**a** the position vector of $P$ after 3 s

**b** the speed of $P$ after 3 s.

**2** In this question use $g = 10\,\text{m}\,\text{s}^{-2}$

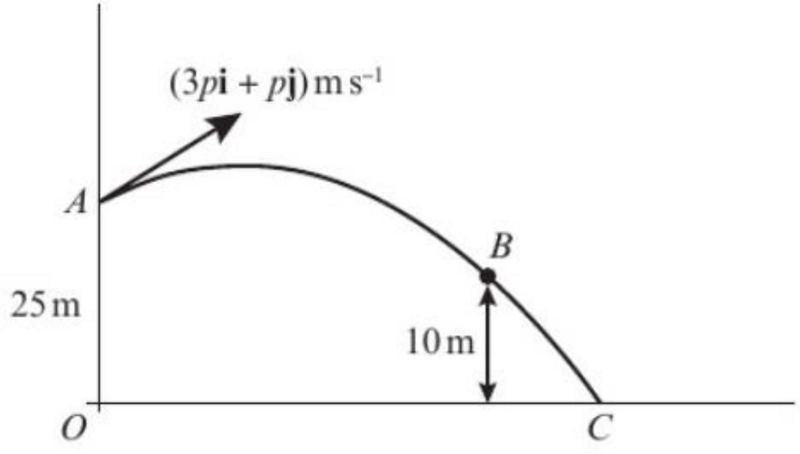
A particle $P$ is projected from the origin with velocity $(4\mathbf{i} + 5\mathbf{j})\,\text{m}\,\text{s}^{-1}$. The particle moves freely under gravity. Find:

**a** the position vector of $P$ after $t$ s

**b** the greatest height of the particle.

**Hint** When the particle is at its greatest height, the **j**-component of the velocity will be 0.

**3** A ball is projected from a point $A$ at the top of a cliff, with position vector $25\mathbf{j}$ m relative to the base of the cliff $O$. The base of the cliff is at sea level. The velocity of projection is $(3p\mathbf{i} + p\mathbf{j})$ m s$^{-1}$, where $p$ is a constant. After 2 seconds, the ball passes a point $B$ with position vector $(q\mathbf{i} + 10\mathbf{j})$ m, where $q$ is a constant, before hitting the sea at point $C$. The ball is modelled as a particle moving freely under gravity and the sea is modelled as a horizontal plane.




**a** Suggest, with reasons, which of these two modelling assumptions is most realistic. **(2 marks)**

**b** Find the velocity vector of the ball at point $B$. **(6 marks)**

A remote-control boat leaves $O$ at the same time the ball is projected, and travels in a straight line towards $C$ with constant acceleration. Given that the ball lands on the boat,

**c** find the acceleration of the boat. **(6 marks)**

**4** A particle $P$ is projected with velocity $(3u\mathbf{i} + 4u\mathbf{j})$ m s$^{-1}$ from a fixed point $O$ on horizontal ground. Given that $P$ strikes the ground at a point 750 m from $O$,

**a** show that $u = 17.5$ **(6 marks)**

**b** calculate the greatest height above the ground reached by $P$ **(3 marks)**

**c** find the angle the direction of motion of $P$ makes with $\mathbf{i}$ when $t = 5$. **(4 marks)**

## Exercise 8C

1 A particle $P$ moves in a straight line. The acceleration, $a$, of $P$ at time $t$ seconds is given by $a = 1 - \sin \pi t \text{ m s}^{-2}$, where $t \geqslant 0$.

When $t = 0$, the velocity of $P$ is $0 \text{ m s}^{-1}$ and its displacement is $0$ m. Find expressions for:

**a** the velocity at time $t$ seconds

**b** the displacement at time $t$ seconds.

**2** A particle moving in a straight line has acceleration $a$, given by

$$a = \sin 3\pi t \text{ m s}^{-2},\ t \geqslant 0$$

At time $t$ seconds the particle has velocity $v$ m s$^{-1}$ and displacement $s$ m. Given that when $t = 0$, $v = \frac{1}{3\pi}$ and $s = 1$, find:

**a** an expression for $v$ in terms of $t$

**b** the maximum speed of the particle

**c** an expression for $s$ in terms of $t$.

**3** An object moves in a straight line from a point $O$. At time $t$ seconds the object has acceleration, $a$, where

$$a = -\cos 4\pi t \text{ m s}^{-2},\ 0 \leqslant t \leqslant 4$$

When $t = 0$, the velocity of the object is 0 m s$^{-1}$ and its displacement is 0 m. Find:

**a** an expression for the velocity at time $t$ seconds

**b** the maximum speed of the object

**c** an expression for the displacement of the object at time $t$ seconds

**d** the maximum distance of the object from $O$

**e** the number of times the object changes direction during its motion.

**Problem-solving**

In part **e**, consider the number of times the velocity changes sign.

## Exercise 8D

**1** At time $t$ seconds, a particle $P$ has position vector $\mathbf{r}$ m with respect to a fixed origin $O$, where

$$\mathbf{r} = (3t - 4)\mathbf{i} + (t^3 - 4t)\mathbf{j},\ t \geqslant 0$$

Find:

**a** the velocity of $P$ when $t = 3$

**b** the acceleration of $P$ when $t = 3$.

**2** A particle $P$ of mass 3 grams moving in a plane is acted on by a force $\mathbf{F}$ N. Its velocity at time $t$ seconds is given by $\mathbf{v} = (t^2\mathbf{i} + (2t - 3)\mathbf{j})\ \text{m s}^{-1},\ t \geqslant 0$.

Find $\mathbf{F}$ when $t = 4$.

## Exercise 8E

**1** A particle $P$ starts from rest at a fixed origin $O$. The acceleration of $P$ at time $t$ seconds (where $t \geqslant 0$) is $(6t^2\mathbf{i} + (8 - 4t^3)\mathbf{j})\,\text{m s}^{-2}$. Find:

**a** the velocity of $P$ when $t = 2$ **(3 marks)**

**b** the position vector of $P$ when $t = 4$. **(3 marks)**

**2** A particle $P$ is moving in a plane with velocity $\mathbf{v}\,\text{m s}^{-1}$ at time $t$ seconds where

$$\mathbf{v} = (3t^2 + 2)\mathbf{i} + (6t - 4)\mathbf{j},\ t \geqslant 0$$

When $t = 2$, $P$ has position vector $9\mathbf{j}$ m with respect to a fixed origin $O$. Find:

**a** the distance of $P$ from $O$ when $t = 0$ **(4 marks)**

**b** the acceleration of $P$ at the instant when it is moving parallel to the vector $\mathbf{i}$. **(4 marks)**

**5** At time $t$ seconds (where $t \geqslant 0$), a particle $P$ is moving in a plane with acceleration $(2\mathbf{i} - 2t\mathbf{j})\,\text{m}\,\text{s}^{-2}$. When $t = 0$, the velocity of $P$ is $2\mathbf{j}\,\text{m}\,\text{s}^{-1}$ and the position vector of $P$ is $6\mathbf{i}\,\text{m}$ with respect to a fixed origin $P$.

**a** Find the position vector of $P$ at time $t$ seconds. **(5 marks)**

At time $t$ seconds (where $t \geqslant 0$), a second particle $Q$ is moving in the plane with velocity $((3t^2 - 4)\mathbf{i} - 2t\mathbf{j})\,\text{m}\,\text{s}^{-1}$. The particles collide when $t = 3$.

**b** Find the position vector of $Q$ at time $t = 0$. **(4 marks)**

**6** At time $t = 0$ a particle $P$ is at rest at a point with position vector $(4\mathbf{i} - 6\mathbf{j})\,\text{m}$ with respect to a fixed origin $O$. The acceleration of $P$ at time $t$ seconds (where $t \geqslant 0$) is $((4t - 3)\mathbf{i} - 6t^2\mathbf{j})\,\text{m}\,\text{s}^{-2}$. Find:

**a** the velocity of $P$ when $t = \frac{1}{2}$ **(5 marks)**

**b** the position vector of $P$ when $t = 6$. **(5 marks)**

**7** At time $t$ seconds (where $t \geqslant 0$) the particle $P$ is moving in a plane with acceleration $\mathbf{a}\,\text{m}\,\text{s}^{-2}$, where $\mathbf{a} = (8t^3 - 6t)\mathbf{i} + (8t - 3)\mathbf{j}$.

When $t = 2$, the velocity of $P$ is $(16\mathbf{i} + 3\mathbf{j})\,\text{m}\,\text{s}^{-1}$. Find:

**a** the velocity of $P$ after $t$ seconds **(4 marks)**

**b** the value of $t$ when $P$ is moving parallel to $\mathbf{i}$. **(3 marks)**

## Exercise 8A

1 **a** $6\mathbf{i} + 12\mathbf{j}$ **b** $-7\mathbf{i} + 4\mathbf{j}$ **c** $-2\mathbf{i} + 6\mathbf{j}$
**d** $10\mathbf{i} - 13\mathbf{j}$ **e** $2\mathbf{i} - 3\mathbf{j}$ **f** 4 s

2 $\frac{\sqrt{85}}{2}\,\mathrm{m\,s^{-1}}$, 319° **3** 2.5 s

4 **a** $\begin{pmatrix} 120 - 30t \\ -10 + 40t \end{pmatrix}$ **b** 4 s

5 $2.03\,\mathrm{m\,s^{-1}}$

6 **a** $7t\mathbf{i} + (400 + 7t)\mathbf{j}$, $(500 - 3t)\mathbf{i} + 15t\mathbf{j}$
**b** $350\mathbf{i} + 750\mathbf{j}$

## Exercise 8B

1 **a** $(36\mathbf{i} + 27.9\mathbf{j})$ m **b** $13\,\text{m s}^{-1}$ (2 s.f.)

2 **a** $\mathbf{r} = (4t)\mathbf{i} + (5t - 5t^2)\mathbf{j}$ **b** 1.25 m

3 **a** Either answer with justification
e.g. The sea is likely to be horizontal and relatively flat, whereas the ball is subject to air resistance, so the assumption that sea is a horizontal plane is most reasonable.
Or e.g. Although the sea is horizontal it is unlikely to be flat because of waves, so the assumption that the ball is a particle is most reasonable.

**b** $\mathbf{v} = (6.9\mathbf{i} - 17\mathbf{j})\,\text{m s}^{-1}$ (both values to 2 s.f.)

**c** $5.5\,\text{m s}^{-2}$ (2 s.f.)

4 **a** $R(\uparrow)$: $0 = 4ut - \frac{g}{2}t^2 \Rightarrow t = \frac{8u}{g}$
$R(\rightarrow)$: $750 = 3ut = \frac{24u^2}{g} \Rightarrow u^2 = \frac{750g}{24} \Rightarrow u = 17.5$

**b** 250m **c** $22°$ (nearest degree)

5 **a** 48 m **b** 120 m (2 s.f.)

**c** $T = 2.5$ s, $\mathbf{r} = (20\mathbf{i} - \frac{45}{8}\mathbf{j})$ m

6 **a** $x = at \Rightarrow t = \frac{x}{a}$
$y = bt - 5t^2 \Rightarrow y = b\left(\frac{x}{a}\right) - 5\left(\frac{x}{a}\right)^2 \Rightarrow y = \frac{bx}{a} - \frac{5x^2}{a^2}$

**b** **i** $X = 1.6b$ **ii** $Y = 0.05b^2$

## Exercise 8C

1 **a** $v = t + \frac{\cos \pi t}{\pi} - \frac{1}{\pi}$ **b** $s = \frac{t^2}{2} + \frac{\sin \pi t}{\pi^2} - \frac{t}{\pi}$

2 **a** $v = -\frac{\cos 3\pi t}{3\pi} + \frac{2}{3\pi}$ **b** $\frac{1}{\pi}$

**c** $s = -\frac{\sin 3\pi t}{9\pi^2} + \frac{2t}{3\pi} + 1$

3 **a** $v = -\frac{\sin 4\pi t}{4\pi}$ **b** $\frac{1}{4\pi}$

**c** $s = \frac{\cos 4\pi t}{16\pi^2} - \frac{1}{16\pi^2}$ **d** $\frac{1}{8\pi^2}$ **e** 15

4 **a** $1.18\,\mathrm{m\,s^{-1}}$ **b** $-0.152\,\mathrm{m\,s^{-2}}$

**c** $-0.759\,\mathrm{N}$

5 **a** $0.5\,\mathrm{m\,s^{-1}}$ **b** $0.1\,\mathrm{m\,s^{-1}}$

6 **a** $12.9\,\mathrm{m\,s^{-1}}$ in the direction of $s$ increasing

**b** $24\,\mathrm{m\,s^{-1}}$ in the direction of $s$ decreasing

**c** 132 m **d** 20.8 m and 118.5 m

## Exercise 8D

1 **a** $(3\mathbf{i} + 23\mathbf{j})\,\mathrm{m\,s^{-1}}$ **b** $18\mathbf{j}\,\mathrm{m\,s^{-2}}$

2 $(0.024\mathbf{i} + 0.006\mathbf{j})\,\mathrm{N}$

## Exercise 8E

1 **a** $16\mathbf{i}\,\mathrm{m\,s^{-1}}$ **b** $128\mathbf{i} - 140.8\mathbf{j}\,\mathrm{m}$

2 **a** $13\,\mathrm{m}$ **b** $(4\mathbf{i} + 6\mathbf{j})\,\mathrm{m\,s^{-2}}$

3 **a** $\mathbf{v} = \left(t^2 - 4t + 2\pi - \frac{\pi^2}{4}\right)\mathbf{i} - 6\cos t\mathbf{j}$ **b** $(2\pi^2 - 4\pi)\,\mathrm{m\,s^{-1}}$

4 **a** $\left(\left(\frac{5t^2}{2} - 3t + 2\right)\mathbf{i} + \left(8t - \frac{t^2}{2} - 5\right)\mathbf{j}\right)\mathrm{m\,s^{-1}}$

**b** $t = \frac{1}{2}$ **c** $\frac{9\sqrt{2}}{8}\,\mathrm{m\,s^{-1}}$

5 **a** $\left((t^2 + 6)\mathbf{i} + \left(2t - \frac{t^3}{3}\right)\mathbf{j}\right)\mathrm{m}$ **b** $6\mathbf{j}\,\mathrm{m}$

6 **a** $(-\mathbf{i} - \frac{1}{4}\mathbf{j})\,\mathrm{m\,s^{-1}}$ **b** $(94\mathbf{i} - 654\mathbf{j})\,\mathrm{m}$

7 **a** $((2t^4 - 3t^2 - 4)\mathbf{i} + (4t^2 - 3t - 7)\mathbf{j})\,\mathrm{m\,s^{-1}}$

**b** $t = \frac{7}{4}$

8 **a** $\mathbf{r} = (2t^2 - 3t + 1)\mathbf{i} + (4t + 2)\mathbf{j}\,\mathrm{m}$

**b** **i** 3.4 **ii** $\mathbf{r} = 36\mathbf{i} + 22\mathbf{j}$