Integrations trig

Exercise 11C

1 Integrate the following:

> **Hint** For part **a**, use $1 + \cot^2 x \equiv \operatorname{cosec}^2 x$. For part **c**, use $\sin 2A \equiv 2\sin A\cos A$, making a suitable substitution for A.

a $\cot^2 x$ **b** $\cos^2 x$

c $\sin 2x \cos 2x$ **d** $(1 + \sin x)^2$

e $\tan^2 3x$ **f** $(\cot x - \operatorname{cosec} x)^2$

g $(\sin x + \cos x)^2$ **h** $\sin^2 x \cos^2 x$ **i** $\dfrac{1}{\sin^2 x \cos^2 x}$ **j** $(\cos 2x - 1)^2$

2 Find the following integrals.

a $\displaystyle\int \frac{1 - \sin x}{\cos^2 x}\,dx$ **b** $\displaystyle\int \frac{1 + \cos x}{\sin^2 x}\,dx$ **c** $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$

d $\displaystyle\int \frac{\cos^2 x}{\sin^2 x}\,dx$ **e** $\displaystyle\int \frac{(1 + \cos x)^2}{\sin^2 x}\,dx$ **f** $\displaystyle\int (\cot x - \tan x)^2\,dx$

g $\displaystyle\int (\cos x - \sin x)^2\,dx$ **h** $\displaystyle\int (\cos x - \sec x)^2\,dx$ **i** $\displaystyle\int \frac{\cos 2x}{1 - \cos^2 2x}\,dx$

3 Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,dx = \frac{2 + \pi}{8}$

4 Find the exact value of each of the following:

a $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\sin^2 x \cos^2 x}\,dx$ **b** $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} (\sin x - \operatorname{cosec} x)^2\,dx$ **c** $\displaystyle\int_0^{\frac{\pi}{4}} \frac{(1 + \sin x)^2}{\cos^2 x}\,dx$ **d** $\displaystyle\int_{\frac{3\pi}{8}}^{\frac{\pi}{2}} \frac{\sin 2x}{1 - \sin^2 2x}\,dx$

5 **a** By expanding $\sin(3x + 2x)$ and $\sin(3x - 2x)$ using the double-angle formulae, or otherwise, show that $\sin 5x + \sin x \equiv 2\sin 3x \cos 2x$.

b Hence find $\displaystyle\int \sin 3x \cos 2x\,dx$

6 $f(x) = 5\sin^2 x + 7\cos^2 x$

a Show that $f(x) = \cos 2x + 6$.

b Hence, find the exact value of $\displaystyle\int_0^{\frac{\pi}{4}} f(x)\,dx$.

7 **a** Show that $\cos^4 x \equiv \frac{1}{8}\cos 4x + \frac{1}{2}\cos 2x + \frac{3}{8}$

b Hence find $\displaystyle\int \cos^4 x\,dx$.

Exercise 11C

1 **a** $-\cot x - x + c$ **b** $\frac{1}{2}x + \frac{1}{4}\sin 2x + c$

c $-\frac{1}{8}\cos 4x + c$ **d** $\frac{3}{2}x - 2\cos x - \frac{1}{4}\sin 2x + c$

e $\frac{1}{3}\tan 3x - x + c$ **f** $-2\cot x - x + 2\operatorname{cosec} x + c$

g $x - \frac{1}{2}\cos 2x + c$ **h** $\frac{1}{8}x - \frac{1}{32}\sin 4x + c$

i $-2\cot 2x + c$ **j** $\frac{3}{2}x + \frac{1}{8}\sin 4x - \sin 2x + c$

2 **a** $\tan x - \sec x + c$ **b** $-\cot x - \operatorname{cosec} x + c$

c $2x - \tan x + c$ **d** $-\cot x - x + c$

e $-2\cot x - x - 2\operatorname{cosec} x + c$

f $-\cot x - 4x + \tan x + c$ **g** $x + \frac{1}{2}\cos 2x + c$

h $-\frac{3}{2}x + \frac{1}{4}\sin 2x + \tan x + c$ **i** $-\frac{1}{2}\operatorname{cosec} 2x + c$

3 $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2 x\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$\displaystyle= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{\pi}{8} + \frac{1}{4} = \frac{2 + \pi}{8}$

4 **a** $\dfrac{4\sqrt{3}}{3}$ **b** $\dfrac{9\sqrt{3} - 10 - \pi}{8}$ **c** $2\sqrt{2} - \dfrac{\pi}{4}$ **d** $\dfrac{\sqrt{2} - 1}{2}$

5 **a** $\sin(3x + 2x) = \sin 3x \cos 2x + \cos 3x \sin 2x$
$\sin(3x - 2x) = \sin 3x \cos 2x - \cos 3x \sin 2x$
Adding gives $\sin 5x + \sin x = 2\sin 3x \cos 2x$

b So $\int \sin 3x \cos 2x\,dx = \int \frac{1}{2}(\sin 5x + \sin x)\,dx$
$= \frac{1}{2}(-\frac{1}{5}\cos 5x - \cos x) + c = -\frac{1}{10}\cos 5x - \frac{1}{2}\cos x + c$

6 **a** $5\sin^2 x + 7\cos^2 x = 5 + 2\cos^2 x = 6 + (2\cos^2 x - 1)$
$= \cos 2x + 6$

b $\frac{1}{2}(1 + 3\pi)$

7 **a** $\cos^4 x = (\cos^2 x)^2 = \left(\dfrac{1 + \cos 2x}{2}\right)^2 = \dfrac{1}{4} + \dfrac{1}{2}\cos 2x$

$+ \dfrac{1}{4}\cos^2 2x = \dfrac{1}{4} + \dfrac{1}{2}\cos 2x + \dfrac{1}{4}\left(\dfrac{1 + \cos 4x}{2}\right)$

$= \dfrac{3}{8} + \dfrac{1}{2}\cos 2x + \dfrac{1}{8}\cos 4x$

b $\dfrac{1}{32}\sin 4x + \dfrac{1}{4}\sin 2x + \dfrac{3}{8}x + c$

Exercise 11D

1 Integrate the following functions.

a $\dfrac{x}{x^2+4}$ **b** $\dfrac{e^{2x}}{e^{2x}+1}$ **c** $\dfrac{x}{(x^2+4)^3}$

d $\dfrac{e^{2x}}{(e^{2x}+1)^3}$ **e** $\dfrac{\cos 2x}{3+\sin 2x}$ **f** $\dfrac{\sin 2x}{(3+\cos 2x)^3}$

g xe^{x^2} **h** $\cos 2x(1+\sin 2x)^4$ **i** $\sec^2 x \tan^2 x$

2 Find the following integrals.

a $\int (x+1)(x^2+2x+3)^4\,dx$ **b** $\int \operatorname{cosec}^2 2x \cot 2x\,dx$

c $\int \sin^5 3x \cos 3x\,dx$ **d** $\int \cos x\, e^{\sin x}\,dx$

e $\int \dfrac{e^{2x}}{e^{2x}+3}\,dx$ **f** $\int x(x^2+1)^{\frac{3}{2}}\,dx$

g $\int (2x+1)\sqrt{x^2+x+5}\,dx$ **h** $\int \dfrac{2x+1}{\sqrt{x^2+x+5}}\,dx$

i $\int \dfrac{\sin x \cos x}{\sqrt{\cos 2x+3}}\,dx$ **j** $\int \dfrac{\sin x \cos x}{\cos 2x+3}\,dx$

3 Find the exact value of each of the following:

a $\int_0^3 (3x^2+10x)\sqrt{x^3+5x^2+9}\,dx$ **b** $\int_{\frac{\pi}{9}}^{\frac{2\pi}{9}} \dfrac{6\sin 3x}{1-\cos 3x}\,dx$

c $\int_4^7 \dfrac{x}{x^2-1}\,dx$ **d** $\int_0^{\frac{\pi}{4}} \sec^2 x\, e^{4\tan x}\,dx$

4 Given that $\int_0^k kx^2 e^{x^3}\,dx = \frac{2}{3}(e^8-1)$, find the value of k.

5 Given that $\int_0^\theta 4\sin 2x \cos^4 2x\,dx = \frac{4}{5}$ where $0 < \theta < \pi$, find the exact value of θ.

6 a By writing $\cot x = \dfrac{\cos x}{\sin x}$, find $\int \cot x\,dx$.

b Show that $\int \tan x\,dx \equiv \ln|\sec x| + c$.

Exercise 11D

1 a $\frac{1}{2}\ln|x^2+4|+c$ **b** $\frac{1}{2}\ln|e^{2x}+1|+c$

c $-\frac{1}{4}(x^2+4)^{-2}+c$ **d** $-\frac{1}{4}(e^{2x}+1)^{-2}+c$

e $\frac{1}{2}\ln|3+\sin 2x|+c$ **f** $\frac{1}{4}(3+\cos 2x)^{-2}+c$

g $\frac{1}{2}e^{x^2}+c$ **h** $\frac{1}{10}(1+\sin 2x)^5+c$

i $\frac{1}{3}\tan^3 x+c$ **j** $\tan x+\frac{1}{3}\tan^3 x+c$

2 a $\frac{1}{10}(x^2+2x+3)^5+c$ **b** $-\frac{1}{4}\cot^2 2x+c$

c $\frac{1}{18}\sin^6 3x+c$ **d** $e^{\sin x}+c$

e $\frac{1}{2}\ln|e^{2x}+3|+c$ **f** $\frac{1}{5}(x^2+1)^{\frac{5}{2}}+c$

g $\frac{2}{3}(x^2+x+5)^{\frac{3}{2}}$ **h** $2(x^2+x+5)^{\frac{1}{2}}+c$

i $-\frac{1}{2}(\cos 2x+3)^{\frac{1}{2}}+c$ **j** $-\frac{1}{4}\ln|\cos 2x+3|+c$

3 a 468 **b** $2\ln 3$ **c** $\frac{1}{2}\ln\left(\frac{16}{5}\right)$ **d** $\frac{1}{4}(e^4-1)$

4 $k=2$ **5** $\theta=\frac{\pi}{2}$

6 a $\ln|\sin x|+c$

b $\int \tan x\,dx = -\ln|\cos x|+c = \ln\left|\frac{1}{\cos x}\right|+c$

$= \ln|\sec x|+c$

Exercise 11E

1 Use the substitutions given to find:

a $\int x\sqrt{1+x}\,\mathrm{d}x$; $u = 1 + x$

b $\int \frac{1+\sin x}{\cos x}\,\mathrm{d}x$; $u = \sin x$

c $\int \sin^3 x\,\mathrm{d}x$; $u = \cos x$

d $\int \frac{2}{\sqrt{x}(x-4)}\,\mathrm{d}x$; $u = \sqrt{x}$

e $\int \sec^2 x \tan x\sqrt{1+\tan x}\,\mathrm{d}x$; $u^2 = 1 + \tan x$

f $\int \sec^4 x\,\mathrm{d}x$; $u = \tan x$

2 Use the substitutions given to find the exact values of:

a $\int_0^5 x\sqrt{x+4}\,\mathrm{d}x$; $u = x + 4$

b $\int_0^2 x(2+x)^3\,\mathrm{d}x$; $u = 2 + x$

c $\int_0^{\frac{\pi}{2}} \sin x\sqrt{3\cos x + 1}\,\mathrm{d}x$; $u = \cos x$

d $\int_0^{\frac{\pi}{3}} \sec x \tan x\sqrt{\sec x + 2}\,\mathrm{d}x$; $u = \sec x$

e $\int_1^4 \frac{1}{\sqrt{x}(4x-1)}\,\mathrm{d}x$; $u = \sqrt{x}$

3 By choosing a suitable substitution, find:

a $\int x(3+2x)^5\,\mathrm{d}x$

b $\int \frac{x}{\sqrt{1+x}}\,\mathrm{d}x$

c $\int \frac{\sqrt{x^2+4}}{x}\,\mathrm{d}x$

4 By choosing a suitable substitution, find the exact values of:

a $\int_2^7 x\sqrt{2+x}\,\mathrm{d}x$

b $\int_2^5 \frac{1}{1+\sqrt{x-1}}\,\mathrm{d}x$

c $\int_0^{\frac{\pi}{2}} \frac{\sin 2\theta}{1+\cos\theta}\,\mathrm{d}\theta$

5 Using the substitution $u^2 = 4x + 1$, or otherwise, find the exact value of $\int_6^{20} \frac{8x}{\sqrt{4x+1}}\,\mathrm{d}x$. **(8 marks)**

6 Use the substitution $u^2 = \mathrm{e}^x - 2$ to show that $\int_{\ln 3}^{\ln 4} \frac{\mathrm{e}^{4x}}{\mathrm{e}^x - 2}\,\mathrm{d}x = \frac{a}{b} + c\ln d$, where a, b, c and d are integers to be found. **(7 marks)**

7 Prove that $-\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arccos x + c$. **(5 marks)**

8 Use the substitution $u = \cos x$ to show

$\int_0^{\frac{\pi}{3}} \sin^3 x \cos^2 x\,\mathrm{d}x = \frac{47}{480}$ **(7 marks)**

Hint Use exact trigonometric values to change the limits in x to limits in u.

9 Using a suitable trigonometric substitution for x, find $\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} x^2\sqrt{1-x^2}\,\mathrm{d}x$. **(8 marks)**

Exercise 11E

1 a $\frac{2}{5}(1+x)^{\frac{5}{2}} - \frac{2}{3}(1+x)^{\frac{3}{2}} + c$ b $-\ln|1 - \sin x| + c$

c $\frac{\cos^3 x}{3} - \cos x + c$ d $\ln\left|\frac{\sqrt{x}-2}{\sqrt{x}+2}\right| + c$

e $\frac{2}{5}(1+\tan x)^{\frac{5}{2}} - \frac{2}{3}(1+\tan x)^{\frac{3}{2}} + c$

f $\tan x + \frac{1}{3}\tan^3 x + c$

2 a $\frac{506}{15}$ b $\frac{392}{5}$ c $\frac{14}{9}$ d $\frac{16}{3} - 2\sqrt{3}$ e $\frac{1}{2}\ln\frac{9}{5}$

3 a $\frac{(3+2x)^7}{28} - \frac{(3+2x)^6}{8} + c$ b $\frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + c$

c $\sqrt{x^2+4} + \ln\left|\frac{\sqrt{x^2+4}-2}{\sqrt{x^2+4}+2}\right| + c$

4 a $\frac{886}{15}$ b $2 + 2\ln\frac{2}{3}$ c $2 - 2\ln 2$

5 $\frac{592}{3}$

9 $\frac{2\pi + 3\sqrt{3}}{96}$

(Integration by Parts)

Exercise 11F

1 Find the following integrals.

a $\int x \sin x \, dx$ b $\int xe^x \, dx$ c $\int x \sec^2 x \, dx$

d $\int x \sec x \tan x \, dx$ e $\int \frac{x}{\sin^2 x} \, dx$

2 Find the following integrals.

a $\int 3 \ln x \, dx$ b $\int x \ln x \, dx$ c $\int \frac{\ln x}{x^3} \, dx$

d $\int (\ln x)^2 \, dx$ e $\int (x^2 + 1) \ln x \, dx$

Hint You will need to use these standard results. In your exam they will be given in the formulae booklet:

- $\int \tan x \, dx = \ln|\sec x| + c$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + c$
- $\int \cot x \, dx = \ln|\sin x| + c$
- $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + c$

3 Find the following integrals.

a $\int x^2 e^{-x} \, dx$ b $\int x^2 \cos x \, dx$ c $\int 12x^2(3 + 2x)^5 \, dx$ d $\int 2x^2 \sin 2x \, dx$ e $\int 2x^2 \sec^2 x \tan x \, dx$

4 Evaluate the following:

a $\int_0^{\ln 2} xe^{2x} \, dx$ b $\int_0^{\frac{\pi}{2}} x \sin x \, dx$ c $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ d $\int_1^2 \frac{\ln x}{x^2} \, dx$

e $\int_0^1 4x(1 + x)^3 \, dx$ f $\int_0^{\pi} x \cos \frac{1}{4} x \, dx$ g $\int_0^{\frac{\pi}{3}} \sin x \ln(\sec x) \, dx$

Exercise 11F

1 a $-x\cos x + \sin x + c$ b $xe^x - e^x + c$
c $x\tan x - \ln|\sec x| + c$ d $x\sec x - \ln|\sec x + \tan x| + c$
e $-x\cot x + \ln|\sin x| + c$

2 a $3x\ln x - 3x + c$ b $\frac{x^2}{2}\ln x - \frac{x^2}{4} + c$
c $-\frac{\ln x}{2x^2} - \frac{1}{4x^2} + c$ d $x(\ln x)^2 - 2x\ln x + 2x + c$
e $\frac{x^3}{3}\ln x - \frac{x^3}{9} + x\ln x - x + c$

3 a $-e^{-x}x^2 - 2xe^{-x} - 2e^{-x} + c$
b $x^2\sin x + 2x\cos x - 2\sin x + c$
c $x^2(3 + 2x)^6 - \frac{x(3 + 2x)^7}{7} + \frac{(3 + 2x)^8}{112} + c$
d $-x^2\cos 2x + x\sin 2x + \frac{1}{2}\cos 2x + c$
e $x^2\sec^2 x - 2x\tan x + 2\ln|\sec x| + c$

4 a $2\ln 2 - \frac{3}{4}$ b 1 c $\frac{\pi}{2} - 1$
d $\frac{1}{2}(1 - \ln 2)$ e 9.8 f $2\sqrt{2}\pi + 8\sqrt{2} - 16$
g $\frac{1}{2}(1 - \ln 2)$