Numerical Mathods

A) Locating Roots

Exercise 10A

1 Show that each of these functions has at least one root in the given interval.

a $f(x) = x^3 - x + 5, -2 < x < -1$

b $f(x) = x^2 - \sqrt{x} - 10, 3 < x < 4$

c $f(x) = x^3 - \frac{1}{x} - 2, -0.5 < x < -0.2$

d $f(x) = e^x - \ln x - 5, 1.65 < x < 1.75$

E 2 $f(x) = 3 + x^2 - x^3$

a Show that the equation $f(x) = 0$ has a root, α, in the interval $[1.8, 1.9]$.

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 1.864$ correct to 3 decimal places.

E 3 $h(x) = \sqrt[3]{x} - \cos x - 1$, where x is in radians.

a Show that the equation $h(x) = 0$ has a root, α, between $x = 1.4$ and $x = 1.5$.

b By choosing a suitable interval, show that $\alpha = 1.441$ is correct to 3 decimal places.

E 4 $f(x) = \sin x - \ln x, x > 0$, where x is in radians.

a Show that $f(x) = 0$ has a root, α, in the interval $[2.2, 2.3]$.

b By considering a change of sign of $f(x)$ in a suitable interval, verify that $\alpha = 2.219$ correct to 3 decimal places.

B) Iteration

Exercise 10B

1 $f(x) = x^2 - 6x + 2$

a Show that $f(x) = 0$ can be written as:

i $x = \dfrac{x^2 + 2}{6}$ **ii** $x = \sqrt{6x - 2}$ **iii** $x = 6 - \dfrac{2}{x}$

b Starting with $x_0 = 4$, use each iterative formula to find a root of the equation $f(x) = 0$. Round your answers to 3 decimal places.

c Use the quadratic formula to find the roots to the equation $f(x) = 0$, leaving your answer in the form $a \pm \sqrt{b}$, where a and b are constants to be found.

2 $f(x) = x^2 - 5x - 3$

a Show that $f(x) = 0$ can be written as:

i $x = \sqrt{5x + 3}$ **ii** $x = \dfrac{x^2 - 3}{5}$

b Let $x_0 = 5$. Show that each of the following iterative formulae gives different roots of $f(x) = 0$.

i $x_{n+1} = \sqrt{5x_n + 3}$ **ii** $x_{n+1} = \dfrac{x_n^2 - 3}{5}$

3 $f(x) = x^2 - 6x + 1$

a Show that the equation $f(x) = 0$ can be written as $x = \sqrt{6x - 1}$.

b Sketch on the same axes the graphs of $y = x$ and $y = \sqrt{6x - 1}$.

c Write down the number of roots of $f(x)$.

d Use your diagram to explain why the iterative formula $x_{n+1} = \sqrt{6x_n - 1}$ converges to a root of $f(x)$ when $x_0 = 2$.

$f(x) = 0$ can also be rearranged to form the iterative formula $x_{n+1} = \dfrac{x_n^2 + 1}{6}$

e By sketching a diagram, explain why the iteration diverges when $x_0 = 10$.

4 $f(x) = xe^{-x} - x + 2$

a Show that the equation $f(x) = 0$ can be written as $x = \ln\left|\dfrac{x}{x - 2}\right|, x \neq 2$.

$f(x)$ has a root, α, in the interval $-2 < x < -1$.

b Use the iterative formula $x_{n+1} = \ln\left|\dfrac{x_n}{x_n - 2}\right|, x \neq 2$ with $x_0 = -1$ to find, to 2 decimal places, the values of x_1, x_2 and x_3.

1 b) i) 0.354 ii) 5.646 iii) 5.646

2b) i) 5.5 ii) −0.5

3c) 2 4b) −1.10, −1.04, −1.07

c) Newton-Raphson

Exercise 10C

1 $f(x) = x^3 - 2x - 1$

a Show that the equation $f(x) = 0$ has a root, α, in the interval $1 < \alpha < 2$.

b Using $x_0 = 1.5$ as a first approximation to α, apply the Newton–Raphson procedure once to $f(x)$ to find a second approximation to α, giving your answer to 3 decimal places.

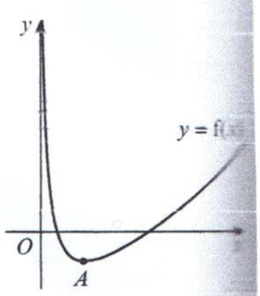
2 $f(x) = x^2 - \frac{4}{x} + 6x - 10,\ x \neq 0.$

a Use differentiation to find $f'(x)$.

The root, α, of the equation $f(x) = 0$ lies in the interval $[-0.4, -0.3]$.

b Taking -0.4 as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.

3 The diagram shows part of the curve with equation $y = f(x)$, where $f(x) = x^{\frac{3}{2}} - e^{-x} + \frac{1}{\sqrt{x}} - 2,\ x > 0.$



The point A, with x-coordinate q, is a stationary point on the curve. The equation $f(x) = 0$ has a root α in the interval $[1.2, 1.3]$.

a Explain why $x_0 = q$ is not suitable to use as a first approximation when applying the Newton–Raphson method. **(1 mark)**

b Taking $x_0 = 1.2$ as a first approximation to α, apply the Newton–Raphson process once to $f(x)$ to obtain a second approximation to α. Give your answer to 3 decimal places.

1) b) 1.632

2 a) $2x + \frac{4}{x^2} + 6$ b) -0.326

3 b) 1.247

Exercise 10D

1 An astronomer is studying the motion of a planet moving along an elliptical orbit. She formulates the following model relating the angle moved at a given time, E radians, to the angle the planet would have moved if it had been travelling on a circular path, M radians:

$$M = E - 0.1 \sin E, \quad E \geqslant 0$$

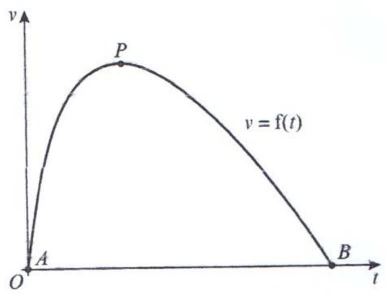
In order to predict the position of the planet at a particular time, the astronomer needs to find the value of E when $M = \frac{\pi}{6}$

a Show that this value of E is a root of the function $f(x) = x - 0.1 \sin x - k$ where k is a constant to be determined.

b Taking 0.6 as a first approximation, apply the Newton–Raphson procedure once to $f(x)$ to obtain a second approximation for the value of E when $M = \frac{\pi}{6}$

c By considering a change of sign on a suitable interval of $f(x)$, show that your answer to part **b** is correct to 3 decimal places.

2 The diagram shows a sketch of part of the curve with equation $v = f(t)$, where $f(t) = \left(10 - \frac{1}{2}(t+1)\right)\ln(t+1)$. The function models the velocity in m/s of a skier travelling in a straight line.



a Find the coordinates of A and B.

b Find $f'(t)$.

c Given that P is a stationary point on the curve, show that the t-coordinate of P lies between 5.8 and 5.9.

d Show that the t-coordinate of P is the solution to

$$t = \frac{20}{1 + \ln(t+1)} - 1$$

An approximation for the t-coordinate of P is found using the iterative formula

$$t_{n+1} = \frac{20}{1 + \ln(t_n + 1)} - 1$$

e Let $t_0 = 5$. Find the values of t_1, t_2 and t_3. Give your answers to 3 decimal places.

1) a) $\pi/6$ b) 0.5781

2 a) (0,0) (19,0)

b) $\frac{10}{t+1} - \frac{\ln(t+1)}{2} - \frac{1}{2}$

e) 6.164, 5.736, 5.897