Differentiation

A: $\sin x$ and $\cos x$

1 Prove, from first principles, that the derivative of $\sin x$ is $\cos x$.

You may assume the formula for $\sin(A + B)$ and that as $h \to 0$, $\dfrac{\sin h}{h} \to 1$ and $\dfrac{\cos h - 1}{h} \to 0$.

2 Differentiate:

a $y = 2\cos x$ **b** $y = 2\sin \frac{1}{2}x$ **c** $y = \sin 8x$ **d** $y = 6\sin \frac{2}{3}x$

3 Find $f'(x)$ given that:

a $f(x) = 2\cos x$ **b** $f(x) = 6\cos \frac{5}{6}x$ **c** $f(x) = 4\cos \frac{1}{2}x$ **d** $f(x) = 3\cos 2x$

4 Find $\dfrac{dy}{dx}$ given that:

a $y = \sin 2x + \cos 3x$ **b** $y = 2\cos 4x - 4\cos x + 2\cos 7x$

c $y = x^2 + 4\cos 3x$ **d** $y = \dfrac{1 + 2x\sin 5x}{x}$

5 A curve has equation $y = x - \sin 3x$. Find the stationary points of the curve in the interval $0 \leqslant x \leqslant \pi$.

6 Find the gradient of the curve $y = 2\sin 4x - 4\cos 2x$ at the point where $x = \dfrac{\pi}{2}$

1 Let $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos h + \cos x\sin h - \sin x}{h}$$

$$= \lim_{h\to 0}\left[\left(\frac{\cos h - 1}{h}\right)\sin x + \left(\frac{\sin h}{h}\right)\cos x\right]$$

Since $\dfrac{\cos h - 1}{h} \to 0$ and $\dfrac{\sin h}{h} \to 1$ the expression inside the limit $\to (0 \times \sin x + 1 \times \cos x)$

So $\displaystyle\lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} = \cos x$

Hence the derivative of $\sin x$ is $\cos x$.

2 **a** $-2\sin x$ **b** $\cos \frac{1}{2}x$

c $8\cos 8x$ **d** $4\sin \frac{2}{3}x$

3 **a** $-2\sin x$ **b** $-5\sin \frac{5}{6}x$

c $-2\sin\left(\frac{1}{2}x\right)$ **d** $-6\sin 2x$

4 **a** $2\cos 2x - 3\sin 3x$ **b** $-8\sin 4x + 4\sin x - 14\sin 7x$

c $2x - 12\sin 3x$ **d** $-\dfrac{1}{x^2} + 10\cos 5x$

5 $(0.41, -0.532), (1.68, 2.63), (2.50, 1.56)$

6 8 **7** $(0.554, 2.24), (2.12, -2.24)$

B Exponentials and Logarithms

1 a Find $\frac{dy}{dx}$ for each of the following:

a $y = 4e^{7x}$ b $y = 3^x$ c $y = \left(\frac{1}{2}\right)^x$ d $y = \ln 5x$

e $y = 4\left(\frac{1}{3}\right)^x$ f $y = \ln(2x^3)$ g $y = e^{3x} - e^{-3x}$ h $y = \frac{(1 + e^x)^2}{e^x}$

2 Find $f'(x)$ given that:

a $f(x) = 3^{4x}$ b $f(x) = \left(\frac{3}{2}\right)^{2x}$ c $f(x) = 2^{4x} + 4^{2x}$ d $f(x) = \frac{2^{7x} + 8^x}{4^{2x}}$

Hint In parts **c** and **d**, rewrite the terms so that they all have the same base and hence can be simplified.

3 Find the gradient of the curve $y = (e^{2x} - e^{-2x})^2$ at the point where $x = \ln 3$.

4 Find the equation of the tangent to the curve $y = 2^x + 2^{-x}$ at the point $\left(2, \frac{17}{4}\right)$.

5 A curve has the equation $y = e^{2x} - \ln x$. Show that the equation of the tangent at the point with x-coordinate 1 is

$$y = (2e^2 - 1)x - e^2 + 1$$

6 A particular radioactive isotope has an activity, R millicuries at time t days, given by the equation $R = 200 \times 0.9^t$. Find the value of $\frac{dR}{dt}$, when $t = 8$.

1 a $28e^{7x}$ b $3^x \ln 3$ c $\left(\frac{1}{2}\right)^x \ln\frac{1}{2}$ d $\frac{1}{x}$

e $4\left(\frac{1}{3}\right)^x \ln\frac{1}{3}$ f $\frac{3}{x}$ g $3e^{3x} + 3e^{-3x}$ h $-e^{-x} + e^x$

2 a $3^{4x} 4 \ln 3$ b $\left(\frac{3}{2}\right)^{2x} 2 \ln\frac{3}{2}$

c $2^{4x} 8 \ln 2$ d $2^{3x} 3 \ln 2 - 2^{-x} \ln 2$

3 323.95 4 $4y = 15 \ln 2(x-2) + 17$

5 $\frac{dy}{dx} = 2e^{2x} - \frac{1}{x}$ At $x = 1$, $y = e^2$, $\frac{dy}{dx} = 2e^2 - 1$

Equation of tangent: $y - e^2 = (2e^2 - 1)(x - 1)$

$\Rightarrow y = (2e^2 - 1)x - 2e^2 + 1 + e^2 \Rightarrow y = (2e^2 - 1)x - e^2 + 1$

6 -9.07 millicuries/day

C The Chain Rule

1 Differentiate:

a $(1 + 2x)^4$ **b** $(3 - 2x^2)^{-5}$ **c** $(3 + 4x)^{\frac{1}{2}}$ **d** $(6x + x^2)^7$

e $\dfrac{1}{3 + 2x}$ **f** $\sqrt{7 - x}$ **g** $4(2 + 8x)^4$ **h** $3(8 - x)^{-6}$

2 Differentiate:

a $e^{\cos x}$ **b** $\cos(2x - 1)$ **c** $\sqrt{\ln x}$ **d** $(\sin x + \cos x)^5$

e $\sin(3x^2 - 2x + 1)$ **f** $\ln(\sin x)$ **g** $2e^{\cos 4x}$ **h** $\cos(e^{2x} + 3)$

3 Given that $y = \dfrac{1}{(4x + 1)^2}$ find the value of $\dfrac{dy}{dx}$ at $\left(\frac{1}{4}, \frac{1}{4}\right)$.

4 A curve C has equation $y = (5 - 2x)^3$. Find the tangent to the curve at the point P with x-coordinate 1.

5 Given that $y = (1 + \ln 4x)^{\frac{3}{2}}$, find the value of $\dfrac{dy}{dx}$ at $x = \dfrac{1}{4}e^3$.

6 Find $\dfrac{dy}{dx}$ for the following curves, giving your answers in terms of y.

a $x = y^2 + y$ **b** $x = e^y + 4y$ **c** $x = \sin 2y$ **d** $4x = \ln y + y^3$

7 Find the value of $\dfrac{dy}{dx}$ at the point $(8, 2)$ on the curve with equation $3y^2 - 2y = x$.

Problem-solving

Your expression for $\dfrac{dy}{dx}$ will be in terms of y. Remember to substitute the y-coordinate into the expression to find the gradient.

8 Find the value of $\dfrac{dy}{dx}$ at the point $\left(\frac{5}{2}, 4\right)$ on the curve with equation $y^{\frac{1}{2}} + y^{-\frac{1}{2}} = x$.

9 **a** Differentiate $e^y = x$ with respect to y.

b Hence, prove that if $y = \ln x$, then $\dfrac{dy}{dx} = \dfrac{1}{x}$

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1 **a** $8(1+2x)^3$ **b** $20x(3-2x^2)^{-6}$
c $2(3 + 4x)^{-\frac{1}{2}}$ **d** $7(6 + 2x)(6x + x^2)^6$
e $-\dfrac{2}{(3 + 2x)^2}$ **f** $-\dfrac{1}{2\sqrt{7 - x}}$
g $128(2 + 8x)^3$ **h** $18(8 - x)^{-7}$

2 **a** $-\sin x\, e^{\cos x}$ **b** $-2\sin(2x - 1)$ **c** $\dfrac{1}{2x\sqrt{\ln x}}$
d $5(\cos x - \sin x)(\sin x + \cos x)^4$
e $(6x - 2)\cos(3x^2 - 2x + 1)$
f $\cot x$ **g** $-8\sin 4x\, e^{\cos 4x}$ **h** $-2e^{2x}\sin(e^{2x} + 3)$

3 -1 **4** $y = -54x + 81$ **5** $12e^{-3}$

6 **a** $\dfrac{1}{2y + 1}$ **b** $\dfrac{1}{e^y + 4}$ **c** $\dfrac{1}{2}\sec 2y$ **d** $\dfrac{4y}{1 + 3y^3}$

7 $\frac{1}{10}$ **8** $\frac{16}{3}$

9 **a** $e^y = \dfrac{dx}{dy}$
b $y = \ln x,\ e^y = x$
Differentiate with respect to y using part **a**
$e^y = \dfrac{dx}{dy} \Rightarrow \dfrac{1}{e^y} = \dfrac{dy}{dx}$
Since $x = e^y$, $\dfrac{dy}{dx} = \dfrac{1}{x}$

D The Product Rule

1 Differentiate:

a $x(1 + 3x)^5$ **b** $2x(1 + 3x^2)^3$ **c** $x^3(2x + 6)^4$ **d** $3x^2(5x - 1)^{-1}$

2 Differentiate:

a $e^{-2x}(2x - 1)^5$ **b** $\sin 2x \cos 3x$ **c** $e^x \sin x$ **d** $\sin (5x) \ln (\mathrm{co}$

3 a Find the value of $\frac{dy}{dx}$ at the point (1, 8) on the curve with equation $y = x^2(3x - 1)^3$.

b Find the value of $\frac{dy}{dx}$ at the point (4, 36) on the curve with equation $y = 3x(2x + 1)^{\frac{1}{2}}$.

c Find the value of $\frac{dy}{dx}$ at the point $\left(2, \frac{1}{5}\right)$ on the curve with equation $y = (x - 1)(2x + 1)^{-1}$

4 Find the stationary points of the curve C with the equation $y = (x - 2)^2(2x + 3)$.

5 A curve C has equation $y = \left(x - \frac{\pi}{2}\right)^5 \sin 2x$, $0 < x < \pi$. Find the gradient of the curve at th point with x-coordinate $\frac{\pi}{4}$

6 A curve C has equation $y = x^2 \cos (x^2)$. Find the equation of the tangent to the curve C at t point $P\left(\frac{\sqrt{\pi}}{2}, \frac{\pi\sqrt{2}}{8}\right)$ in the form $ax + by + c = 0$ where a, b and c are exact constants.

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1 **a** $(3x + 1)^4(18x + 1)$ **b** $2(3x^2 + 1)^2(21x^2 + 1)$
c $16x^2(x + 3)^3(7x + 9)$ **d** $3x(5x - 2)(5x - 1)^{-2}$

2 **a** $-4(x - 3)(2x - 1)^4 e^{-2x}$
b $2 \cos 2x \cos 3x - 3 \sin 2x \sin 3x$
c $e^x(\sin x + \cos x)$ **d** $5 \cos 5x \ln (\cos x) - \tan x \sin 5x$

3 **a** 52 **b** 13 **c** $\frac{3}{25}$

4 $(2, 0), \left(-\frac{1}{3}, \frac{343}{27}\right)$ **5** $\frac{5\pi^4}{256}$

6 $\sqrt{2\pi}(\pi - 4)x + 8y - \pi\sqrt{2}\left(\frac{\pi - 2}{2}\right) = 0$

E The Quotient Rule

1 Differentiate:

a $\dfrac{5x}{x+1}$ **b** $\dfrac{2x}{3x-2}$ **c** $\dfrac{x+3}{2x+1}$ **d** $\dfrac{3x^2}{(2x-1)^2}$ **e** $\dfrac{6x}{(5x+3)^{\frac{1}{2}}}$

2 Differentiate:

a $\dfrac{e^{4x}}{\cos x}$ **b** $\dfrac{\ln x}{x+1}$ **c** $\dfrac{e^{-2x}+e^{2x}}{\ln x}$ **d** $\dfrac{(e^x+3)^3}{\cos x}$ **e** $\dfrac{\sin^2 x}{\ln x}$

3 Find the value of $\dfrac{dy}{dx}$ at the point $\left(1, \frac{1}{4}\right)$ on the curve with equation $y = \dfrac{x}{3x+1}$

4 Find the value of $\dfrac{dy}{dx}$ at the point (12, 3) on the curve with equation $y = \dfrac{x+3}{(2x+1)^{\frac{1}{2}}}$

5 Find the stationary points of the curve C with equation $y = \dfrac{e^{2x+3}}{x}$, $x \neq 0$.

6 Find the equation of the tangent to the curve $y = \dfrac{e^{\frac{1}{3}x}}{x}$ at the point $\left(3, \frac{1}{3}e\right)$.

7 Find the exact value of $\dfrac{dy}{dx}$ at the point $x = \dfrac{\pi}{9}$ on the curve with equation $y = \dfrac{\ln x}{\sin 3x}$

8 The curve C has equation $x = \dfrac{e^y}{3+2y}$

a Find the coordinates of the point P where the curve cuts the x-axis.

b Find an equation of the normal to the curve at P, giving your answer in the form $y = mx + c$, where m and c are integers to be found.

1 a $\dfrac{5}{(x+1)^2}$ **b** $-\dfrac{4}{(3x-2)^2}$ **c** $-\dfrac{5}{(2x+1)^2}$

d $-\dfrac{6x}{(2x-1)^3}$ **e** $\dfrac{15x+18}{(5x+3)^{\frac{3}{2}}}$

2 a $\dfrac{e^{4x}(\sin x + 4\cos x)}{\cos^2 x}$ **b** $\dfrac{1}{x(x+1)} - \dfrac{\ln x}{(x+1)^2}$

c $\dfrac{e^{-2x}((2xe^{4x} - 2x)\ln x - e^{4x} - 1)}{x(\ln x)^2}$

d $\dfrac{(e^x+3)^2((e^x+3)\sin x + 3e^x\cos x)}{\cos^2 x}$

e $\dfrac{2\sin x\cos x}{\ln x} - \dfrac{\sin^2 x}{x(\ln x)^2}$

3 $\dfrac{1}{16}$ **4** $\dfrac{2}{25}$ **5** $(0.5, 2e^4)$

6 $y = \frac{1}{3}e$ **7** $\dfrac{6\sqrt{3} - 2\pi\ln\left(\frac{\pi}{9}\right)}{\pi}$

8 a $\left(\frac{1}{3}, 0\right)$ **b** $y = -\frac{1}{9}x + \frac{1}{27}$

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F All Trig Functions

1 Differentiate:

a $y = \tan 3x$ **b** $y = 4\tan^3 x$ **c** $y = \tan(x - 1)$ **d** $y = x^2 \tan \frac{1}{2}x + \tan\left(x - \frac{1}{2}\right)$

2 Differentiate:

a $\cot 4x$ **b** $\sec 5x$ **c** $\operatorname{cosec} 4x$ **d** $\sec^2 3x$

e $x \cot 3x$ **f** $\dfrac{\sec^2 x}{x}$ **g** $\operatorname{cosec}^3 2x$ **h** $\cot^2(2x - 1)$

3 Find the function $f'(x)$ where $f(x)$ is:

a $(\sec x)^{\frac{1}{2}}$ **b** $\sqrt{\cot x}$ **c** $\operatorname{cosec}^2 x$ **d** $\tan^2 x$

e $\sec^3 x$ **f** $\cot^3 x$

4 Find $f'(x)$ where $f(x)$ is:

a $x^2 \sec 3x$ **b** $\dfrac{\tan 2x}{x}$ **c** $\dfrac{x^2}{\tan x}$ **d** $e^x \sec 3x$

e $\dfrac{\ln x}{\tan x}$ **f** $\dfrac{e^{\tan x}}{\cos x}$

5 The curve C has equation

$$y = \frac{1}{\cos x \sin x}, \quad 0 < x \leqslant \pi$$

a Find $\dfrac{dy}{dx}$

b Determine the number of stationary points of the curve C.

c Find the equation of the tangent at the point where $x = \frac{\pi}{3}$, giving your answer in the form $ax + by + c = 0$, where a, b and c are exact constants to be determined.

1 **a** $3\sec^2 3x$ **b** $12\tan^2 x \sec^2 x$ **c** $\sec^2(x - 1)$

d $\frac{1}{2}x^2 \sec^2 \frac{1}{2}x + 2x \tan \frac{1}{2}x + \sec^2\left(x - \frac{1}{2}\right)$

2 **a** $-4\operatorname{cosec}^2 4x$ **b** $5\sec 5x \tan 5x$

c $-4\operatorname{cosec} 4x \cot 4x$ **d** $6\sec^2 3x \tan 3x$

e $\cot 3x - 3x \operatorname{cosec}^2 3x$ **f** $\dfrac{\sec^2 x(2x\tan x - 1)}{x^2}$

g $-6\operatorname{cosec}^3 2x \cot 2x$

h $-4\cot(2x - 1)\operatorname{cosec}^2(2x - 1)$

3 **a** $\frac{1}{2}(\sec x)^{\frac{1}{2}} \tan x$ **b** $-\frac{1}{2}(\cot x)^{-\frac{1}{2}} \operatorname{cosec}^2 x$

c $-2\operatorname{cosec}^2 x \cot x$ **d** $2\tan x \sec^2 x$

e $3\sec^3 x \tan x$ **f** $-3\cot^2 x \operatorname{cosec}^2 x$

4 **a** $2x\sec 3x + 3x^2 \sec 3x \tan 3x$

b $\dfrac{2x\sec^2 2x - \tan 2x}{x^2}$ **c** $\dfrac{2x\tan x - x^2\sec^2 x}{\tan^2 x}$

d $e^x \sec 3x(1 + 3\tan 3x)$ **e** $\dfrac{\tan x - x\sec^2 x \ln x}{x\tan^2 x}$

f $e^{\tan x} \sec x(\tan x + \sec^2 x)$

5 **a** $\dfrac{1}{\cos^2 x} - \dfrac{1}{\sin^2 x}$ **b** 2

c $24x - 9y + 12\sqrt{3} - 8\pi = 0$

G Parametric Differentiation

1 Find $\frac{dy}{dx}$ for each of the following, leaving your answer in terms of the parameter t.

a $x = 2t$, $y = t^2 - 3t + 2$
b $x = 3t^2$, $y = 2t^3$
c $x = t + 3t^2$, $y = 4t$
d $x = t^2 - 2$, $y = 3t^5$
e $x = \frac{2}{t}$, $y = 3t^2 - 2$
f $x = \frac{1}{2t-1}$, $y = \frac{t^2}{2t-1}$
g $x = \frac{2t}{1+t^2}$, $y = \frac{1-t^2}{1+t^2}$
h $x = t^2e^t$, $y = 2t$
i $x = 4\sin 3t$, $y = 3\cos 3t$
j $x = 2 + \sin t$, $y = 3 - 4\cos t$
k $x = \sec t$, $y = \tan t$
l $x = 2t - \sin 2t$, $y = 1 - \cos 2t$
m $x = e^t - 5$, $y = \ln t$, $t > 0$
n $x = \ln t$, $y = t^2 - 64$, $t > 0$
o $x = e^{2t} + 1$, $y = 2e^t - 1$, $-1 < t < 1$

2 **a** Find the equation of the tangent to the curve with parametric equations $x = 3 - 2\sin t$, $y = t\cos t$, at the point P, where $t = \pi$.

b Find the equation of the tangent to the curve with parametric equations $x = 9 - t^2$, $y = t^2 + 6t$, at the point P, where $t = 2$.

3 **a** Find the equation of the normal to the curve with parametric equations $x = e^t$, $y = e^t + e^{-t}$, at the point P, where $t = 0$.

b Find the equation of the normal to the curve with parametric equations $x = 1 - \cos 2t$, $y = \sin 2t$, at the point P, where $t = \frac{\pi}{6}$

10 The curve C has parametric equations

$$x = 2\sin t, \quad y = \sqrt{2}\cos 2t, \quad 0 < t < \pi$$

a Find an expression for $\frac{dy}{dx}$ in terms of t.

The point A lies on C where $t = \frac{\pi}{3}$. The line l is the normal to C at A.

b Find an equation for l in the form $ax + by + c = 0$, where a, b and c are exact constants to be found.

c Prove that the line l does not intersect the curve anywhere other than at point A.

1 **a** $\frac{2t-3}{2}$ **b** $\frac{6t^2}{6t} = t$ **c** $\frac{4}{1+6t}$ **d** $\frac{15t^3}{2}$
e $-3t^3$ **f** $t(1-t)$ **g** $\frac{2t}{t^2-1}$ **h** $\frac{2}{(t^2+2t)e^t}$
i $-\frac{3}{4}\tan 3t$ **j** $4\tan t$ **k** $\operatorname{cosec} t$ **l** $\cot t$
m $\frac{1}{te^t}$ **n** $2t^2$ **o** $\frac{1}{e^t}$

2 **a** $y = -\frac{1}{2}x + \frac{3}{2} - \pi$ **b** $2y + 5x = 57$

3 **a** $x = 1$ **b** $y + \sqrt{3}x = \sqrt{3}$

10 **a** $-2\sqrt{2}\sin t$ **b** $x - \sqrt{6}y - 2\sqrt{3} = 0$

c $2\sin t - \sqrt{12}\cos 2t - 2\sqrt{3} = 0$
$\sin t - \sqrt{3}\cos 2t - \sqrt{3} = 0$
$2\sqrt{3}\sin^2 t + \sin t - 2\sqrt{3} = 0$
$(2\sin t - \sqrt{3})(\sqrt{3}\sin t + 2) = 0$
$\sin t = \frac{\sqrt{3}}{2}$ $\left(\text{sint } t \neq -\frac{2}{\sqrt{3}}\right) \Rightarrow t = \frac{\pi}{3}$ or $\frac{2\pi}{3}$
B is when $t = \frac{2\pi}{3}$: $\left(2\sin\frac{2\pi}{3}, \sqrt{2}\cos\frac{4\pi}{3}\right) = \left(\sqrt{3}, -\frac{1}{\sqrt{2}}\right)$
Same point as A, so l only intersects C once.

H Implicit Differentiation

3 Find an expression in terms of x and y for $\frac{dy}{dx}$, given that:

a $x^2 + y^3 = 2$ b $x^2 + 5y^2 = 14$ c $x^2 + 6x - 8y + 5y^2 = 13$

d $y^3 + 3x^2y - 4x = 0$ e $3y^2 - 2y + 2xy = x^3$ f $x = \frac{2y}{x^2 - y}$

g $(x - y)^4 = x + y + 5$ h $e^x y = xe^y$ i $\sqrt{xy} + x + y^2 = 0$

4 Find the equation of the tangent to the curve with implicit equation $x^2 + 3xy^2 - y^3 = 9$ at the point (2, 1).

5 Find the equation of the normal to the curve with implicit equation $(x + y)^3 = x^2 + y$ at the point (1, 0).

6 Find the coordinates of the points of zero gradient on the curve with implicit equation $x^2 + 4y^2 - 6x - 16y + 21 = 0$.

Problem-solving

Find $\frac{dy}{dx}$ then set the numerator equal to 0 to find the x-coordinate at the points of 0 gradient. You need to find two corresponding y-coordinates.

7 A curve C is described by the equation

$$2x^2 + 3y^2 - x + 6xy + 5 = 0$$

Find an equation of the tangent to C at the point (1, −2), giving your answer in the form $ax + by + c = 0$, where a, b and c are integers.

3 a $-\frac{2x}{3y^2}$ b $-\frac{x}{5y}$ c $\frac{-3-x}{5y-4}$

d $\frac{4-6xy}{3x^2+3y^2}$ e $\frac{3x^2-2y}{6y-2+2x}$ f $\frac{3x^2-y}{2+x}$

g $\frac{4(x-y)^3-1}{1+4(x-y)^3}$ h $\frac{e^x y - e^y}{xe^y - e^x}$ i $\frac{-2\sqrt{xy}-y}{4y\sqrt{xy}+x}$

4 $y = -\frac{7}{9}x + \frac{23}{9}$ 5 $y = 2x - 2$

6 (3, 1) and (3, 3) 7 $3x + 2y + 1 = 0$

1 Second Derivatives

1 For each of the following functions, find the interval on which the function is:

i convex **ii** concave

a $f(x) = x^3 - 3x^2 + x - 2$ **b** $f(x) = x^4 - 3x^3 + 2x - 1$ **c** $f(x) = \sin x, 0 < x <$

d $f(x) = -x^2 + 3x - 7$ **e** $f(x) = e^x - x^2$ **f** $f(x) = \ln x, x > 0$

2 $f(x) = \arcsin x, -1 < x < 1$

a Show that $f(x)$ is concave on the interval $(-1, 0)$.

b Show that $f(x)$ is convex on the interval $(0, 1)$.

c Hence deduce the point of inflection of f.

3 Find any point(s) of inflection of the following functions.

a $f(x) = \cos^2 x - 2\sin x, 0 < x < 2\pi$ **b** $f(x) = -\dfrac{x^3 - 2x^2 + x - 1}{x - 2}, x \neq 2$

c $f(x) = -\dfrac{x^3}{x^2 - 4}, x \neq \pm 2$ **d** $f(x) = \arctan x$

4 $f(x) = 2x^2 \ln x, x > 0$

Show that f has exactly one point of inflection and determine the value of x at this poir

5 The curve C has equation $y = e^x(x^2 - 2x + 2)$.

a Find the exact coordinates of the stationary point on C and determine its nature.

b Find the coordinates of any non-stationary points of inflection on C.

1 **a** **i** $(1, \infty)$ **ii** $(-\infty, 1)$

b **i** $(-\infty, 0) \cup \left(\frac{3}{2}, \infty\right)$ **ii** $\left(0, \frac{3}{2}\right)$

c **i** $(\pi, 2\pi)$ **ii** $(0, \pi)$

d **i** nowhere **ii** $(-\infty, \infty)$

e **i** $(\ln 2, \infty)$ **ii** $(-\infty, \ln 2)$

f **i** nowhere **ii** $(0, \infty)$

2 **a** $f'(x) = \dfrac{1}{\sqrt{1 - x^2}}, f''(x) = \dfrac{x}{(1 - x^2)^{\frac{3}{2}}}$

$f''(x) \leqslant 0 \Rightarrow x \leqslant 0$, so $f(x)$ concave for $x \in (-1, 0)$

b $f''(x) \geqslant 0 \Rightarrow x \geqslant 0$, so $f(x)$ convex for $x \in (0, 1)$

c $(0, 0)$

3 **a** $\left(\frac{\pi}{6}, -\frac{1}{4}\right), \left(\frac{5\pi}{6}, -\frac{1}{4}\right)$

b $(1, -1)$ **c** $(0, 0)$ **d** $(0, 0)$

4 $f'(x) = 2x + 4x\ln x = 2x(1 + 2\ln x), f''(x) = 6 + 4\ln x$

$f''(x) = 0 \Rightarrow 4\ln x = -6 \Rightarrow \ln x = -\frac{3}{2} \Rightarrow x = e^{-\frac{3}{2}}$

There is one point of inflection where $x = e^{-\frac{3}{2}}$

5 **a** $(0, 2)$, point of inflection **b** $\left(-2, \frac{10}{e^2}\right)$

J Rates of Change

1 Given that $A = \frac{1}{4}\pi r^2$ and that $\frac{dr}{dt} = 6$, find $\frac{dA}{dt}$ when $r = 2$.

2 Given that $y = xe^x$ and that $\frac{dx}{dt} = 5$, find $\frac{dy}{dt}$ when $x = 2$.

3 Given that $r = 1 + 3\cos\theta$ and that $\frac{d\theta}{dt} = 3$, find $\frac{dr}{dt}$ when $\theta = \frac{\pi}{6}$

4 Given that $V = \frac{1}{3}\pi r^3$ and that $\frac{dV}{dt} = 8$, find $\frac{dr}{dt}$ when $r = 3$.

5 A population is growing at a rate which is proportional to the size of the population. Write down a differential equation for the growth of the population.

6 A curve C has equation $y = f(x)$, $y > 0$. At any point P on the curve, the gradient of C is proportional to the product of the x- and the y-coordinates of P. The point A with coordinate (4, 2) is on C and the gradient of C at A is $\frac{1}{2}$

Show that $\frac{dy}{dx} = \frac{xy}{16}$

7 Liquid is pouring into a container at a constant rate of $30\,\text{cm}^3\,\text{s}^{-1}$. At time t seconds liquid is leaking from the container at a rate of $\frac{2}{15}V\,\text{cm}^3\,\text{s}^{-1}$, where $V\,\text{cm}^3$ is the volume of the liquid in the container at that time.

Show that $-15\frac{dV}{dt} = 2V - 450$.

8 An electrically-charged body loses its charge, Q coulombs, at a rate, measured in coulombs per second, proportional to the charge Q.
Write down a differential equation in terms of Q and t where t is the time in seconds since the body started to lose its charge.

9 The ice on a pond has a thickness x mm at a time t hours after the start of freezing. The rate of increase of x is inversely proportional to the square of x.
Write down a differential equation in terms of x and t.

10 The radius of a circle is increasing at a constant rate of 0.4 cm per second.

a Find $\frac{dC}{dt}$, where C is the circumference of the circle, and interpret this value in the context of the model.

b Find the rate at which the area of the circle is increasing when the radius is 10 cm.

c Find the radius of the circle when its area is increasing at the rate of $20\,\text{cm}^2$ per second.

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1 6π 2 $15e^2$ 3 $-\frac{9}{2}$ 4 $\frac{8}{9\pi}$ 5 $\frac{dP}{dt} = kP$

6 $\frac{dy}{dx} = kxy$; at (4, 2) $\frac{dy}{dx} = \frac{1}{2}$, so $8k = \frac{1}{2}$, $k = \frac{1}{16}$

Therefore $\frac{dy}{dx} = \frac{xy}{16}$

7 $\frac{dV}{dt}$ = rate in − rate out $= 30 - \frac{2}{15}V \Rightarrow 15\frac{dV}{dt} = 450 - 2V$

So $-15\frac{dV}{dt} = 2V - 450$

8 $\frac{dQ}{dt} = -kQ$ 9 $\frac{dx}{dt} = \frac{k}{x^2}$

10 **a** Circumference, C, $= 2\pi r$, so $\frac{dC}{dt} = 2\pi \times 0.4$
$= 0.8\pi\,\text{cm}\,\text{s}^{-1}$
Rate of increase of circumference with respect to time.

b $8\pi\,\text{cm}^2\,\text{s}^{-1}$ **c** $\frac{25}{\pi}$ cm