2nd Year Assignment 21

1. Find $\frac{dy}{dx}$ in terms of *x* and *y* for each of the following

a. $e^{2x} + e^{2y} = xy$,

b. $e^{x^2} \sin y + 5 \ln x = 6$

c. $(1 - x^2)(3 - \sec y) = 5xy$

d. $y2^x - \tan y = x^3y^6$

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2. At time *t* seconds, the surface area of a cube is $A\ cm^2$ and the volume is V cm^3.
The surface area of the cube is expanding at a constant rate of 2 cm^2s^{-1}

a. Write an expression for *V* in terms of *A*.

b. Find an expression for $\frac{dV}{dt}$

c. Show that $\frac{dV}{dt} = kV^n$, where k and n are numerical constants to be stated.

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3. The amount of a certain type of drug in the bloodstream *t* hours after it has been taken is given by the formula $x = De^{-\frac{1}{8}t}$, where *x* is the amount of the drug in the bloodstream in milligrams and *D* is the dose given in milligrams.

A dose of 10 mg of the drug is given.

(*a*) Find the amount of the drug in the bloodstream 5 hours after the dose is given. Give your answer in mg to 3 decimal places.

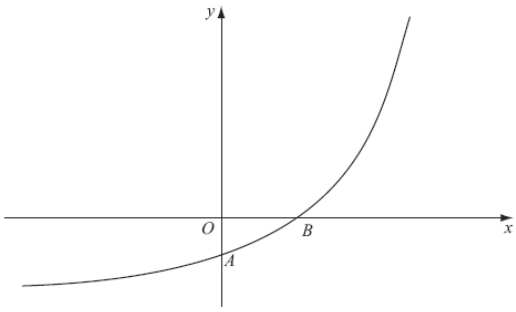
A second dose of 10 mg is given after 5 hours.

(*b*) Show that the amount of the drug in the bloodstream 1 hour after the second dose is 13.549 mg to 3 decimal places.

No more doses of the drug are given. At time *T* hours after the second dose is given, the amount of the drug in the bloodstream is 3 mg.

(*c*) Find the value of *T*.

4. The graph shows a sketch of part of the curve with equation $y = \mathrm{f}(x)$, $x \in \mathbb{R}$. The curve meets the coordinate axes at the points $A(0, 1 - k)$ and $B(\frac{1}{2} \ln k, 0)$, where k is a constant and $k > 1$.



On separate diagrams, sketch the curve with equation

(*a*) $y = |\mathrm{f}(x)|$,

(*b*) $y = \mathrm{f}^{-1}(x)$.

Show on each sketch the coordinates, in terms of k, of each point at which the curve meets or cuts the axes.

Given that $\mathrm{f}(x) = \mathrm{e}^{2x} - k$,

(*c*) state the range of f,

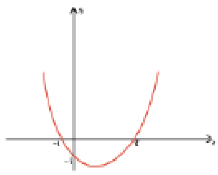
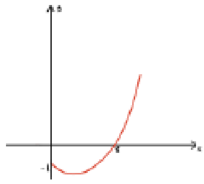
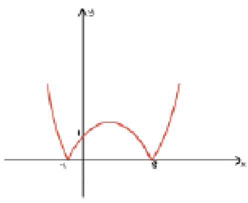
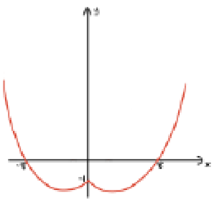
(*d*) find $\mathrm{f}^{-1}(x)$,

(*e*) write down the domain of f^{-1}.

HINT: b) The graph of $f^{-1}(x)$ is the graph of $y = f(x)$ reflected in $y = x$

c) The minimum value of e^{2x} is 0. Work out the minimum value of $e^{2x} - k$ and the range will be all values greater than this.

Example 2: Given that $f(x) = x^2 - 7x - 8$, sketch: (i) $|f(x)|$, (ii) $f(|x|)$

(i) $\lvert f(x) \rvert$		(ii) $f(\lvert x \rvert)$	
Step	Corresponding graph	Step	Corresponding graph
We start by sketching $y = x^2 - 7x - 8$.		Start by sketching $y = x^2 - 7x - 8$ for $x \geq 0$.	
Next, we reflect the portion of the graph below the x-axis in the x-axis.		Now reflect the graph in the y-axis.	

Example 3: Find the inverse of the function $f(x) = x^3 - 8$ $\{x \in \mathbb{R}, x \geq 2\}$, stating its range.

[1] We have $y = x^3 - 8$, so interchanging x and y:	$x = y^3 - 8$
[2] Rearranging to make y the subject	$y = \sqrt[3]{x + 8}$
[3] This function is our inverse. The range will be the same as the domain of the original function	$f^{-1}(x) = \sqrt[3]{x + 8}$ Range: $y \geq 2$

5. A group of office workers were questioned for a health magazine and $\frac{2}{5}$ were found to take regular exercise. When questioned about their eating habits $\frac{2}{3}$ said they always eat breakfast and, of those who always eat breakfast $\frac{9}{25}$ also took regular exercise.

i) Find the probability that a randomly selected member of the group
(a) always eats breakfast and takes regular exercise,
(b) does not always eat breakfast and does not take regular exercise.

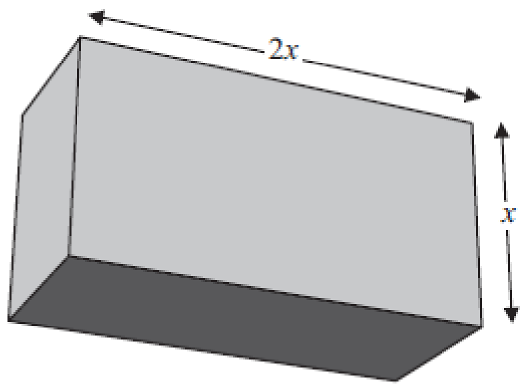
ii) Determine, giving your reason, whether or not always eating breakfast and taking regular exercise are statistically independent.

HINT: Draw a two way table

	ercise	**o exercise**	
eakfast	p_1		$\frac{2}{3}$
o breakfast			
	$\frac{2}{5}$		1

To work out p_1, use the fact that of the $\frac{2}{3}$ who always eat breakfast $\frac{9}{25}$ also took regular exercise. So $p_1 = \frac{9}{25} \text{x} \frac{2}{3}$

6. A cuboid has a rectangular cross-section where the length of the rectangle is equal to twice its width, *x* cm, as shown in the diagram. The volume of the cuboid is 81 cubic centimetres.



(*a*) Show that the total length, *L* cm, of the twelve edges of the cuboid is given by

$$L = 12x + \frac{162}{x^2}.$$

(*b*) Use calculus to find the minimum value of *L*.

(*c*) Justify, by further differentiation, that the value of *L* that you have found is a minimum.

7. (a) Express 3 sin x + 2 cos x in the form R sin (x + α) where R > 0 and $0 < \alpha < \frac{\pi}{2}$.

(b) Hence find the greatest value of $(3 \sin x + 2 \cos x)^4$.

(c) Solve, for $0 < x < 2\pi$, the equation 3 sin x + 2 cos x = 1, giving your answers to 3 decimal places.

8. (a) Prove that $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$ = 2 cosec 2θ, θ ≠ 90n°.

(b) Sketch the graph of y = 2 cosec 2θ for 0° < θ < 360°.

(c) Solve, for 0° < θ < 360°, the equation $\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$ = 3,

giving your answers to 1 decimal place.

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9. A curve is described by the equation $x^3 - 4y^2 = 12xy$.

(*a*) Find the coordinates of the two points on the curve where $x = -8$.

(*b*) Find the gradient of the curve at each of these points.

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10. (a) Use the binomial theorem to expand $(2 - 3x)^{-2}$ in ascending powers of *x*, up to and including the term in x^3. Give each coefficient as a simplified fraction.

$f(x) = \frac{a+bx}{(2-3x)^2}$, where *a* and *b* are constants.

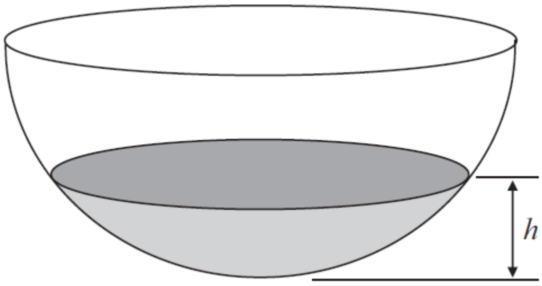
In the binomial expansion of f (x), in ascending powers of *x*, the coefficient of *x* is 0 and the coefficient of x^2 is $\frac{9}{16}$

(b) Find the value of *a* and the value of *b*,

(c) Find the coefficient of x^3, giving your answer as a simplified fraction.

Test Yourself

Give yourself 20 minutes to answer these questions



A) A hollow hemispherical bowl is shown in the diagram. Water is flowing into the bowl.

When the depth of the water is *h* m, the volume $V\ m^3$ is given by

$V = \frac{1}{12}\pi h^2(3 - 4h), \quad 0 \leq h \leq 0.25$

a) Find in terms of $\pi, \frac{dV}{dh}$ when $h = 0.1$

b) Water flows into the bowl at a rate of $\frac{\pi}{800}\ m^3 s^{-1}$. Find the rate of change of h in ms^{-1}, when $h = 0..1$

B) Find the gradient of the curve with equation $ln\ y\ =\ 2x\ ln\ x, \quad x > 0, y > 0$ at the point on the curve where *x* = 2. Give your answer as an exact value.