2nd Year Assignment 17

1. Conor uses a 3D printer to produce various pieces for a model. He records the times taken, t hours, to produce each piece, and its base area, $x\ cm$.

Base area, x (cm^2)	1.1	1.3	1.9	2.2	2.5	3.7
Time, t (hours)	0.7	0.9	1.5	1.8	2.2	3.8

a) Calculate the product moment correlation coefficient between $\log x$ and $log\ t$.

b) Use your answer to part (a) to explain why an equation of the form $t = ax^n$, where a and n are constants, is likely to be a good model for the relationship between x and t.

c) The regression line of $\log t$ on $\log x$ is given as $\log t = -0.210 + 1.38 \log x$. Determine the value of the constants a and n in the equation given in part (b).

2. A new antibiotic is tested by spraying it on a lab dish covered in bacteria. Initially, 12000 bacteria were placed in the dish and 24 hours later, this number had fallen to 2000. The number of bacteria on this lab dish reduced according to the equation $N = Ae^{-kt}, \quad t \geq 0,$ where t is the time in hours since the bacteria were first placed on the dish and A and k are positive constants.

a) Show that $k = 0.07466$, correct to four decimal places.

b) Find the value of t when the bacteria will reach 1000

3 a) Give the binomial expansion of $(1 + x)^{\frac{1}{2}}$ up to and including the term in x^3

b) By substituting $x = \frac{1}{4}$, find an approximation to $\sqrt{5}$ as a fraction.

4) Find the range for each of the following functions.

a) $f(x) = (x - 4)^2 + 1, \quad x \in \mathcal{R}, x > 4$

b) $g(x) = (x + 3)^2 - 1, \quad x \in \mathcal{R}, x \geq -4$

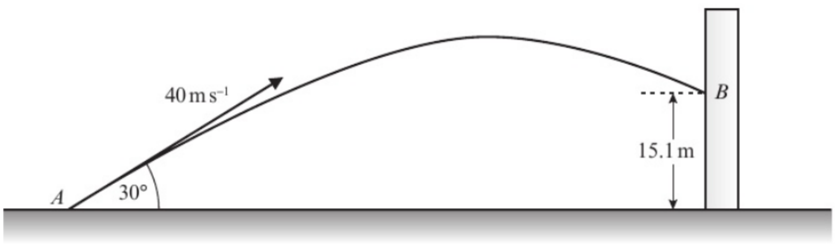
c) $h(x) = (x - 5)^2 + 2, \quad x \in \mathcal{R}, 0 < x < 6$

5. Sue has two coins. One is fair, with a head on one side and a tail on the other.

The second is a trick coin and has a tail on both sides. Sue picks up one of the coins at random and flips it.

a) Find the probability that it lands heads up
b) Given that it lands tails up, find the probability that she picked up the fair coin

6. A golf ball is driven from a point A with a speed of $40\ ms^{-1}$ at an angle of elevation of $30°$. On its downwards flight, the ball hits an advertising hoarding at a height of $15.1\ m$ above the level of A, as shown in the diagram.



Find
a) the time taken by the ball to reach its greatest height above A
b) the time taken by the ball to travel from A to B
c) the speed with which the ball hits the hoarding

7. The curve C has the equation $f(x) = (x - a)(x + b),\quad x \in \mathbb{R}$, where a and b are constants and $a > b > 0$.

Sketch, in separate sets of axes, the graph of

a) $y = f(x)$
b) $y = -f(x + a)$

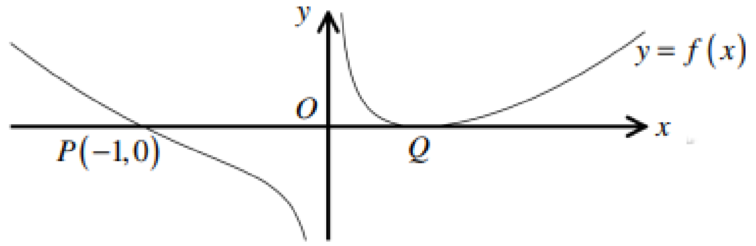
Each of the graphs must show clearly,

i) the coordinates of any points where the curve meets the coordinates axes.
ii) the equation of the line of symmetry of the curve

8. Andrew and Bethany are preparing for a Mathematics exam by doing the same set of practice papers. They both have one practice paper left to do and their mean scores are identical. Andrew scores 83% on his last paper and his mean score is rises to 72% . Bethany scores 47% on her last paper and her mean score is drops to 69% . Determine the number of practice papers in the set.

9. A train travelling at constant speed, takes 14 seconds to cross a bridge of length 240 metres and 6 seconds to go past a lamp post. Determine the speed and the length of the train.

10. The figure shows a curve with equation $y = f(x)$.



The curve meets the x axis at the points P(−1,0) and Q , and its gradient function is given by

$$f'(x) = \frac{8x^3 - 1}{x^2}, \quad x \neq 0$$

a) Find an equation of the tangent to the curve at P
b) Find an expression for $f''(x)$
c) Determine
   i) The equation of the curve
   ii) The coordinates of Q

Formula test

Try to answer all of these in 5 minutes. (We haven't covered Questions 30-46 yet)

1) $ax^2 + bx + c = 0$. What are the roots of this equation? $x =$..

2) $a^x a^y \equiv$...................   3) $a^x \div a^y \equiv$.....................   4) $(a^x)^y \equiv$..........................

5) $x = a^n \Leftrightarrow n =$.................

6) $\log_a x + \log_a y \; \equiv$ ....   7) $\log_a x - \log_a y \; \equiv$...................8) $k \log_a x \; \equiv$.........................

9) A straight line graph, gradient m passing through (x_1, y_1)has equation ....

10) Straight lines with gradients m_1 and m_2 are perpendicular when ....

11) General term of an arithmetic progression, $u_n =$..

12) General term of a geometric progression: $u_n =$..

13) The sine rule is..

14) The cosine rule is ....

15) The area of a triangle is ....

16) $\cos^2 A + \sin^2 A =$ ....   17) $\sec^2 A =$....................   18) $\mathrm{cosec}^2 \mathrm{A} =$.......................

19) $\cos 2A =$..........................   20) $\cos 2A =$..........................   21) $\cos 2A =$..........................

22) $\sin 2A =$ ....   23) $\tan 2A =$......................................

24) The circumference of a circle = ... 25) The area of a circle = ...

26) Pythagoras' theorem is .... 27) The area of a trapezium = ...

28) Volume of a prism = ……………………………….

29) For a circle of radius r, where an angle at the centre of θ radians subtends an arc of length s and encloses an associated sector of area A

i) $s =$………………….. ii) $A =$……………………..

~~Differentiate the following 30) x^n…………………… 31) $\sin kx$……………………~~

~~32) $\cos kx$…………………… 33) e^{kx}…………………… 34) $\ln x$……………………~~

~~35) $f(x) + g(x)$…………………… 36) $f(x)g(x)$ …………………… 37) $f(g(x))$ ……………………~~

~~Integrate the following 38) x^n…………………… 39) $cos\ kx$……………………~~

~~40) $\sin kx$…………………… 41) e^{kx}…………………… 42) $\frac{1}{x}$……………………~~

~~43)$f'(x) + g'(x)$ …………………… 44) $f\big(g(x)\big)g'(x)$ ………………………..~~

~~45) The area under a curve = …………………… 46) $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}|$ = …………………………………~~

47) The mean of a set of data = ………………………..48) The standard Normal variable Z = ……………………

49) Weight = …………………… 50) Friction F…………….. 51) Newton’s Second Law………………………….

52) For motion in a straight line with variable acceleration:

i) v = …………………… ii) a = ……………………

iii) r=…………………… iv) v=……………………

Here are the formulae that you are given in the formula book:

AS Mathematics

Pure Mathematics

Mensuration

Surface area of sphere $= 4\pi r^2$

Area of curved surface of cone $= \pi r \times$ slant height

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Statistics

Probability

$$\mathrm{P}(A') = 1 - \mathrm{P}(A)$$

Standard deviation

Standard deviation = √(Variance)

Interquartile range = $\mathrm{IQR} = \mathrm{Q}_3 - \mathrm{Q}_1$

For a set of n values $x_1, x_2, \ldots x_i, \ldots x_n$

$$\mathrm{S}_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

Standard deviation = $\sqrt{\frac{\mathrm{S}_{xx}}{n}}$ or $\sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$

Statistical tables

The following statistical tables are required for A Level Mathematics:

Binomial Cumulative Distribution Function (see page 29)

Random Numbers (see page 38)

Mechanics

Kinematics

For motion in a straight line with constant acceleration:

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$s = vt - \frac{1}{2}at^2$

$v^2 = u^2 + 2as$

$s = \frac{1}{2}(u + v)t$

2 A Level Mathematics

Pure Mathematics

Mensuration

Surface area of sphere = $4\pi r^2$

Area of curved surface of cone = $\pi r \times$ slant height

Arithmetic series

$$S_n = \tfrac{1}{2}n(a + l) = \tfrac{1}{2}n[2a + (n-1)d]$$

Binomial series

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n \quad (n \in \mathbb{N})$$

where $\binom{n}{r} = {}^nC_r = \dfrac{n!}{r!(n-r)!}$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\times 2}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{1\times 2\times \ldots \times r}x^r + \ldots \quad (|x| < 1, n \in \mathbb{R})$$

Logarithms and exponentials

$$\log_a x = \frac{\log_b x}{\log_b a}$$

$$e^{x \ln a} = a^x$$

Geometric series

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_\infty = \frac{a}{1-r} \text{ for } |r| < 1$$

Trigonometric identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \quad (A \pm B \neq (k + \tfrac{1}{2})\pi)$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$\cos A + \cos B = 2\cos\frac{A+B}{2}\cos\frac{A-B}{2}$$

$$\cos A - \cos B = -2\sin\frac{A+B}{2}\sin\frac{A-B}{2}$$

Small angle approximations

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\tan\theta \approx \theta$$

where θ is measured in radians

Differentiation

First Principles

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$\mathbf{f(x)}$	$\mathbf{f'(x)}$
$\tan kx$	$k\sec^2 kx$
$\sec kx$	$k\sec kx \tan kx$
$\cot kx$	$-k\,\mathrm{cosec}^2\, kx$
$\mathrm{cosec}\, kx$	$-k\,\mathrm{cosec}\, kx \cot kx$
$\dfrac{f(x)}{g(x)}$	$\dfrac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$

Integration (+ constant)

$\mathbf{f(x)}$	$\int \mathbf{f(x)\,dx}$
$\sec^2 kx$	$\frac{1}{k}\tan kx$
$\tan kx$	$\frac{1}{k}\ln\lvert\sec kx\rvert$
$\cot kx$	$\frac{1}{k}\ln\lvert\sin kx\rvert$
$\mathrm{cosec}\, kx$	$-\frac{1}{k}\ln\lvert\mathrm{cosec}\, kx + \cot kx\rvert, \quad \frac{1}{k}\ln\lvert\tan(\frac{1}{2}kx)\rvert$
$\sec kx$	$\frac{1}{k}\ln\lvert\sec kx + \tan kx\rvert, \quad \frac{1}{k}\ln\lvert\tan(\frac{1}{2}kx + \frac{1}{4}\pi)\rvert$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

Numerical Methods

The trapezium rule: $\int_a^b y \, dx \approx \frac{1}{2}h\{(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\}$, where $h = \frac{b-a}{n}$

The Newton-Raphson iteration for solving $f(x) = 0 : x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Statistics

Probability

$$P(A') = 1 - P(A)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A)P(B \mid A)$$

$$P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B \mid A)P(A) + P(B \mid A')P(A')}$$

For independent events A and B,

$$P(B \mid A) = P(B)$$
$$P(A \mid B) = P(A)$$
$$P(A \cap B) = P(A)\,P(B)$$

Standard deviation

Standard deviation = √(Variance)

Interquartile range = $\mathrm{IQR} = \mathrm{Q}_3 - \mathrm{Q}_1$

For a set of n values $x_1, x_2, \ldots x_i, \ldots x_n$

$$S_{xx} = \Sigma(x_i - \bar{x})^2 = \Sigma x_i^2 - \frac{(\Sigma x_i)^2}{n}$$

Standard deviation = $\sqrt{\dfrac{S_{xx}}{n}}$ or $\sqrt{\dfrac{\sum x^2}{n} - \bar{x}^2}$

Discrete distributions

Distribution of X	$\mathrm{P}(X = x)$	Mean	Variance
Binomial $\mathrm{B}(n, p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$

Sampling distributions

For a random sample of n observations from $\mathrm{N}(\mu, \sigma^2)$

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim \mathrm{N}(0, 1)$$

Mechanics

Kinematics

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