Parametric Equations

Exercise 8A

1 Find a Cartesian equation for each of these parametric equations, giving your answer in the form $y = f(x)$. In each case find the domain and range of $f(x)$.

a $x = t - 2, \quad y = t^2 + 1, \quad -4 \leqslant t \leqslant 4$

b $x = 5 - t, \quad y = t^2 - 1, \quad t \in \mathbb{R}$

c $x = \frac{1}{t}, \quad y = 3 - t, \quad t \neq 0$

Notation If the domain of t is given as $t \neq 0$, this implies that t can take any value in $\mathbb{R}$ other than 0.

d $x = 2t + 1, \quad y = \frac{1}{t}, \quad t > 0$

e $x = \frac{1}{t - 2}, \quad y = t^2, \quad t > 2$

f $x = \frac{1}{t + 1}, \quad y = \frac{1}{t - 2}, \quad t > 2$

2 For each of these parametric curves:

i find a Cartesian equation for the curve in the form $y = f(x)$ giving the domain on which the curve is defined

ii find the range of $f(x)$.

a $x = 2\ln(5 - t), \quad y = t^2 - 5, \quad t < 4$

b $x = \ln(t + 3), \quad y = \frac{1}{t + 5}, \quad t > -2$

c $x = e^t, \quad y = e^{3t}, \quad t \in \mathbb{R}$

Exercise 8B

1 Find the Cartesian equation of the curves given by the following parametric equations:

a $x = 2\sin t - 1, \quad y = 5\cos t + 4, \quad 0 < t < 2\pi$

b $x = \cos t, \quad y = \sin 2t, \quad 0 < t < 2\pi$

c $x = \cos t, \quad y = 2\cos 2t, \quad 0 < t < 2\pi$

d $x = \sin t, \quad y = \tan 2t, \quad 0 < t < \frac{\pi}{2}$

e $x = \cos t + 2, \quad y = 4\sec t, \quad 0 < t < \frac{\pi}{2}$

f $x = 3\cot t, \quad y = \operatorname{cosec} t, \quad 0 < t < \pi$

9 Show that the curve with parametric equations

$$x = 2\cos t, \quad y = \sin\left(t - \frac{\pi}{6}\right), \quad 0 < t < \pi$$

can be written in the form

$$y = \frac{1}{4}\left(\sqrt{12 - 3x^2} - x\right), -2 < x < 2$$

10 A curve has parametric equations

$$x = \tan^2 t + 5, \quad y = 5\sin t, \quad 0 < t < \frac{\pi}{2}$$

a Find the Cartesian equation of the curve in the form $y^2 = \mathrm{f}(x)$.

b Determine the possible values of x and y in the given domain of t.

Exercise 8C

1 A curve is given by the parametric equations

$$x = 2t, \quad y = \frac{5}{t}, \quad t \neq 0$$

Copy and complete the table and draw a graph of the curve for $-5 \leqslant t \leqslant 5$.

t	−5	−4	−3	−2	−1	−0.5	0.5	1	2	3	4	5
$x = 2t$	−10	−8				−1						
$y = \frac{5}{t}$	−1	−1.25					10					

2 A curve is given by the parametric equations

$$x = t^2, \quad y = \frac{t^3}{5}$$

Copy and complete the table and draw a graph of the curve for $-4 \leqslant t \leqslant 4$.

t	−4	−3	−2	−1	0	1	2	3	4
$x = t^2$	16								
$y = \frac{t^3}{5}$	−12.8								

4 Sketch the curves given by these parametric equations:

a $x = t - 2, \quad y = t^2 + 1, \quad -4 \leqslant t \leqslant 4$

b $x = 3\sqrt{t}, \quad y = t^3 - 2t, \quad 0 \leqslant t \leqslant 2$

c $x = t^2, \quad y = (2 - t)(t + 3), \quad -5 \leqslant t \leqslant 5$

d $x = 2\sin t - 1, \quad y = 5\cos t + 1, \quad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{4}$

e $x = \sec^2 t - 3, \quad y = 2\sin t + 1, \quad -\frac{\pi}{4} \leqslant t \leqslant \frac{\pi}{2}$

f $x = t - 3\cos t, \quad y = 1 + 2\sin t, \quad 0 \leqslant t \leqslant 2\pi$

Exercise 8D

intersection should be positive.

1 Find the coordinates of the point(s) where the following curves meet the x-axis.

a $x = 5 + t, \quad y = 6 - t$

b $x = 2t + 1, \quad y = 2t - 6$

c $x = t^2, \quad y = (1 - t)(t + 3)$

d $x = \frac{1}{t}, \quad y = (t - 1)(2t - 1), \quad t \neq 0$

e $x = \frac{2t}{1 + t}, \quad y = t - 9, \quad t \neq -1$

2 Find the coordinates of the point(s) where the following curves meet the y-axis.

a $x = 2t, \quad y = t^2 - 5$

b $x = 3t - 4, \quad y = \frac{1}{t^2}, \quad t \neq 0$

c $x = t^2 + 2t - 3, \quad y = t(t - 1)$

d $x = 27 - t^3, \quad y = \frac{1}{t - 1}, \quad t \neq 1$

e $x = \frac{t - 1}{t + 1}, \quad y = \frac{2t}{t^2 + 1}, \quad t \neq -1$

3 A curve has parametric equations $x = 4at^2, \quad y = a(2t - 1)$, where a is a constant. The curve passes through the point (4, 0). Find the value of a.

8 Find the coordinates of the point(s) where the following curves meet the x-axis and the y-axis.

a $x = t^2 - 1, \quad y = \cos t, \quad 0 < t < \pi$

b $x = \sin 2t, \quad y = 2\cos t + 1, \quad \pi < t < 2\pi$

c $x = \tan t, \quad y = \sin t - \cos t, \quad 0 < t < \frac{\pi}{2}$

9 Find the coordinates of the point(s) where the following curves meet the x-axis and the y-axis.

a $x = e^t + 5, \quad y = \ln t, \quad t > 0$

b $x = \ln t, \quad y = t^2 - 64, \quad t > 0$

c $x = e^{2t} + 1, \quad y = 2e^t - 1, \quad -1 < t < 1$

10 Find the values of t at the points of intersection of the line $y = -3x + 2$ and the curve with parametric equations $x = t^2$, $y = t$, and give the coordinates of these points.

11 Find the value(s) of t at the point of intersection of the line $y = x - \ln 3$ and the curve with parametric equations $x = \ln(t - 1)$, $y = \ln(2t - 5)$, $t > \frac{5}{2}$, and give the exact coordinates of this point.

Exercise 8E

1 A river flows from north to south. The position at time t seconds of a rowing boat crossing the river from west to east is modelled by the parametric equations

$$x = 0.9t\text{ m}, \quad y = -3.2t\text{ m}$$

where x is the distance travelled east and y is the distance travelled north.

Given that the river is 75 m wide,

a find the time taken to get to the other side

b find the distance the boat has been moved off-course due to the current

c show that the motion of the boat is a straight line

d determine the speed of the boat.

2 The position of a small plane coming into land at time t seconds after it has started its descent is modelled by the parametric equations

$$x = 80t, \quad y = -9.1t + 3000, \quad 0 \leqslant t < 330$$

where x is the horizontal distance travelled (in metres) and y is the vertical distance (in metres) of the plane above ground level.

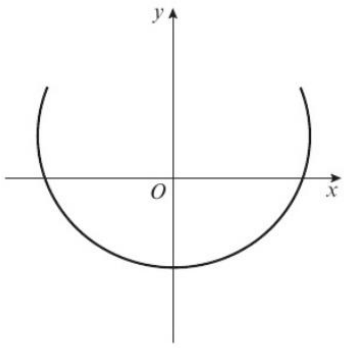
a Find the initial height of the plane.

b Justify the choice of domain, $0 \leqslant t < 330$, for this model.

c Find the horizontal distance the plane travels between beginning its descent and landing.

6 The cross-section of a bowl design is given by the following parametric equations

$$x = t - 4\sin t, \quad y = 1 - 2\cos t, \quad -\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}$$

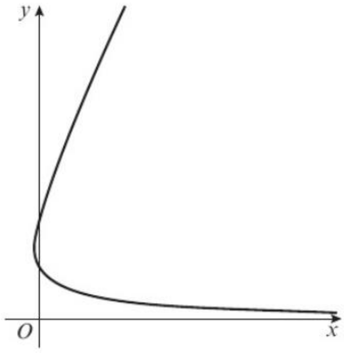


a Find the length of the opening of the bowl. **(3 marks)**

b Given that the cross-section of the bowl crosses the y-axis at its deepest point, find the depth of the bowl. **(4 marks)**

7 A particle is moving in the xy-plane such that its position after time t seconds relative to a fixed origin O is given by the parametric equations

$$x = \frac{t^2 - 3t + 2}{t}, \quad y = 2t, \quad t > 0$$



The diagram shows the path of the particle.

a Find the distance from the origin to the particle at time $t = 0.5$.

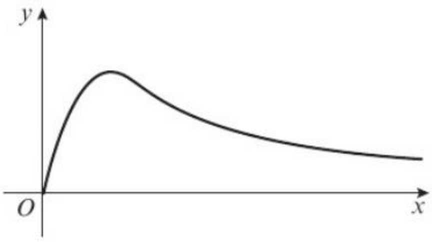
b Find the coordinates of the points where the particle crosses the y-axis.

Another particle travels in the same plane with its path given by the equation $y = 2x + 10$.

c Show that the paths of these two particles never intersect.

9 The profile of a hill climb in a bike race is modelled by the following parametric equations

$$x = 50\tan t \text{ m}, \quad y = 20\sin 2t \text{ m}, \quad 0 < t \leqslant \frac{\pi}{2}$$



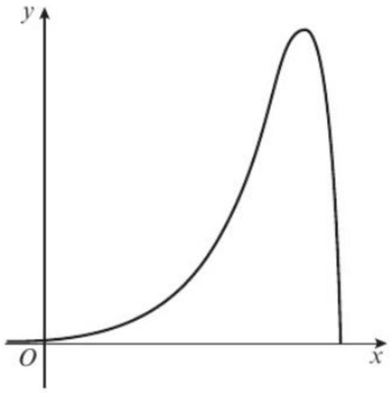
a Find the value of t at the highest point of the hill climb.

b Hence find the coordinates of the highest point.

c Find the coordinates when $t = 1$ and show that at this point, a cyclist will be descending.

10 A computer model for the shape of the path of a rollercoaster is given by the parametric equations

$$x = 5 + \ln t, \quad y = 5\sin 2t, \quad 0 < t \leqslant \frac{\pi}{2}$$



a Find the coordinates of the point where $t = \frac{\pi}{6}$ **(2 marks)**

Given that one unit on the model represents 5 m in real life,

b find the maximum height of the rollercoaster **(1 mark)**

c find the horizontal distance covered during the descent of the rollercoaster. **(4 marks)**

d Hence, find the average gradient of the descent. **(1 mark)**

Exercise 8A

1 a $y = (x + 2)^2 + 1, -6 \leqslant x \leqslant 2,$ $1 \leqslant y \leqslant 17$

b $y = (5 - x)^2 - 1, x \in \mathbb{R}$ $y \geqslant -1$

c $y = 3 - \frac{1}{x}, x \neq 0,$ $y \neq 3$

d $y = \frac{2}{x - 1}, x > 1,$ $y > 0$

e $y = \left(\frac{1 + 2x}{x}\right)^2, x > 0,$ $y > 4$

f $y = \frac{x}{1 - 3x}, 0 < x < \frac{1}{3},$ $y > 0$

2 a i $y = 20 - 10e^{\frac{1}{2}x} + e^x, x > 0$ ii $y \geqslant -5$

b i $y = \frac{1}{e^x + 2}, x > 0$ ii $0 < y < \frac{1}{3}$

c i $y = x^3, x > 0$ ii $y > 0$

Exercise 8B

1 a $25(x + 1)^2 + 4(y - 4)^2 = 100$ b $y^2 = 4x^2(1 - x^2)$

c $y = 4x^2 - 2$ d $y = \frac{2x\sqrt{1 - x^2}}{1 - 2x^2}$

e $y = \frac{4}{x - 2}$ f $y^2 = 1 + \left(\frac{x}{3}\right)^2$

9 $y = \sin t \cos\left(\frac{\pi}{6}\right) - \cos t \sin\left(\frac{\pi}{6}\right)$

$$= \frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t = \frac{\sqrt{3\left(1 - \frac{x^2}{4}\right)}}{2} - \frac{1}{4}x$$

$$= \frac{1}{4}\left(2\sqrt{3 - \frac{3}{4}x^2} - x\right) = \frac{1}{4}(\sqrt{12 - 3x^2} - x)$$

$t = 0 \Rightarrow x = 2, t = \pi \Rightarrow x = -2$, so $-2 < x < 2$.

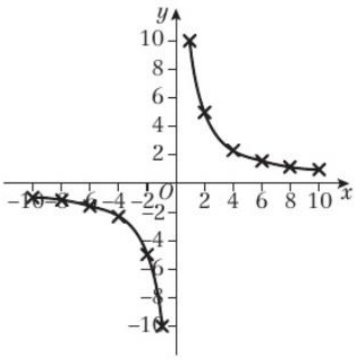
10 a $y^2 = 25\left(1 - \frac{1}{x - 4}\right)$ b $x > 5, 0 < y < 5$

Exercise 8C

1

t	–5	–4	–3	–2	–1	–0.5
$x = 2t$	–10	–8	–6	–4	–2	–1
$y = \frac{5}{t}$	–1	–1.25	–1.67	–2.5	–5	–10

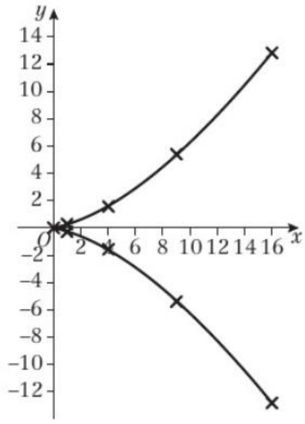
t	0.5	1	2	3	4	5
$x = 2t$	1	2	4	6	8	10
$y = \frac{5}{t}$	10	5	2.5	1.67	1.25	1



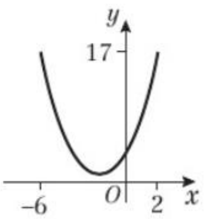
2

t	–4	–3	–2	–1	0
$x = t^2$	16	9	4	1	0
$y = \frac{t^3}{5}$	–12.8	–5.4	–1.6	–0.2	0

t	1	2	3	4
$x = t^2$	1	4	9	16
$y = \frac{t^3}{5}$	0.2	1.6	5.4	12.8



4 **a**



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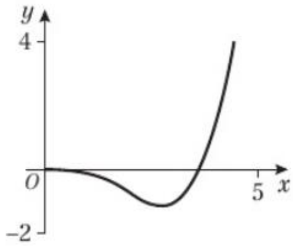

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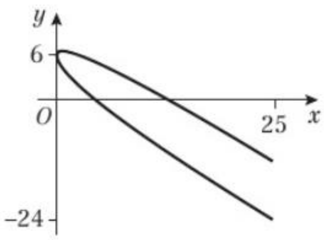

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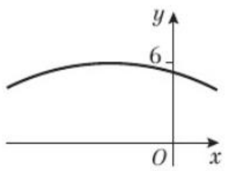

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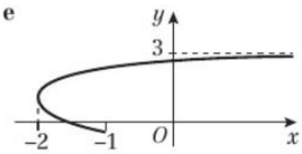

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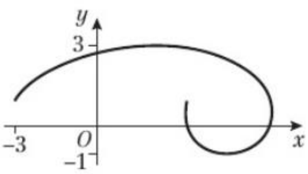

Exercise 8D

1 **a** $(11, 0)$ **b** $(7, 0)$ **c** $(1, 0), (9, 0)$
d $(1, 0), (2, 0)$ **e** $\left(\frac{9}{5}, 0\right)$

2 **a** $(0, -5)$ **b** $\left(0, \frac{9}{16}\right)$ **c** $(0, 0), (0, 12)$
d $\left(0, \frac{1}{2}\right)$ **e** $(0, 1)$

3 4 **4** 4 **5** $\left(\frac{1}{2}, \frac{3}{2}\right)$

6 $t = \frac{5}{2}, t = -\frac{3}{2}; \left(\frac{25}{4}, 5\right), \left(\frac{9}{4}, -3\right)$

7 $(1, 2), (1, -2), (4, 4), (4, -4)$

8 **a** $\left(\frac{\pi^2}{4} - 1, 0\right), (0, \cos 1)$
b $\left(\frac{\sqrt{3}}{2}, 0\right), (0, 1)$
c $(1, 0)$

9 **a** $(e + 5, 0)$ **b** $(\ln 8, 0), (0, -63)$ **c** $\left(\frac{5}{4}, 0\right)$

10 $t = \frac{2}{3}, t = -1, \left(\frac{4}{9}, \frac{2}{3}\right), (1, -1)$

Exercise 8E

1 **a** 83.3 seconds **b** 267 m

c $t = \frac{x}{0.9} \Rightarrow y = -3.2\,\frac{x}{0.9} \Rightarrow y = -\frac{32}{9}x$

which is in the form, $y = mx + c$ and is therefore a straight line.

d $3.32\,\text{ms}^{-1}$

2 **a** 3000 m

b Initial point is when $t = 0$. For $t \geqslant 330$, y is negative ie. the plane is underground or below sea level.

c 26 400 m (3 s.f.)

6 **a** 4.86 (3 s.f.) **b** Depth = 2

7 **a** $\frac{\sqrt{13}}{2}$ **b** (0, 2), (0, 4)

c $2t = 2\left(\frac{t^2 - 3t + 2}{t}\right) + 10$

$2t^2 = 2t^2 - 6t + 4 + 10t \Rightarrow 0 = 4t + 4 \Rightarrow t = -1$

Since, $t > 0$, the paths do not intersect.

8 **a** 10 m

b $k = 1.89$ (3 s.f.). Therefore, time taken is 1.89 seconds.

c 34.1 m (3 s.f.)

d $t = \frac{x}{18}$

$y = -4.9\left(\frac{x}{18}\right)^2 + 4\left(\frac{x}{18}\right) + 10 = -\frac{49}{3240}x^2 + \frac{2}{9}x + 10$

Therefore, the ski jumper's path is a quadratic equation. Maximum height = 10.8 m (3 s.f.)

9 **a** $t = \frac{\pi}{4}$ **b** (50, 20)

c (77.87, 18.19)

$\frac{\pi}{4} < 1 < \frac{\pi}{2}$, which is when $\sin 2t$ is decreasing, hence when y is decreasing, hence the cyclist is descending.

10 **a** (4.35, 4.33) (3 s.f.) **b** 25 m

c 3.47 m (3 s.f.) **d** −7.21