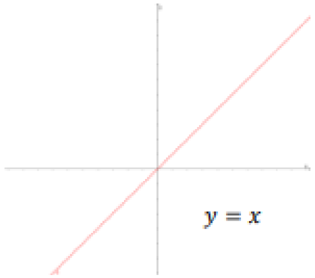
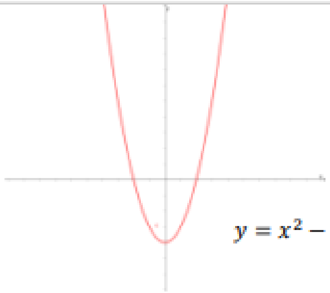
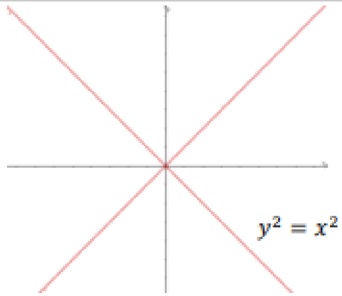
Functions and Graphs Cheat Sheet

What really is a function?

Before we can properly define a function, we must first understand mappings.

A mapping takes a set of numbers and transforms them into another set of numbers. Mappings are often represented by algebraic equations. For example, $y = x^2$ is a mapping that takes every real number and maps them to their respective squares.

There are three different types of mappings you need to be well-acquainted with:

Mapping type	**One-to-one**	**Many-to-one**	**One-to-many**
Description	Each input has a unique output.	Each input has just one associated output, but there are some inputs that are mapped to the same output.	There exist inputs that are mapped to more than one output.
Example			

- Functions are mappings that are one-to-one or many-to-one. One-to-many mappings are not functions!

Function notation

You need to understand what is meant by the domain and range of a function:

- The domain of a function is the set of all possible inputs (i.e. the set of x-values the function takes)
- The range of a function is the set of all possible outputs (i.e. the set of y-values the function takes)

There are two main types of notation you will see for denoting functions:

- $f(x) = \sqrt{x}$, $\{\boldsymbol{x \in \mathbb{R}, x \geq 0}\}$
- $f: x \mapsto \sqrt{x}$, $\{\boldsymbol{x \in \mathbb{R}, x \geq 0}\}$

These notations are equivalent for one-to-one or many-to-one functions. The "$\mapsto$" symbol means that each element in the domain is mapped to just one element in the range. The bold text represents the domain of the function.

Composition of functions

We can combine two or more functions to make a new function. This new function is called a composite function.

Let's take two functions, $g(x)$ and $f(x)$. If we wish to combine them, there are two ways we can do this:

- $fg(x)$ we interpret this as applying $g(x)$ first, then $f(x)$.
- $gf(x)$ we interpret this as applying $f(x)$ first, then $g(x)$.

It is important to realise that $gf(x) \neq fg(x)$ in general.

When composing functions, it is helpful to interpret $fg(x)$ as $f[g(x)]$. This means that we must replace the inputs of $f(x)$ with $g(x)$, to get $fg(x)$.

Below is an example of how we do this in practice.

Example 1: Given that $f(x) = 3x - 2$ and $g(x) = x^2 + 4x - 2$, find an expression for:
(i) $fg(x)$
(ii) $gf(x)$

(i) $fg(x)$ is found by replacing all the x terms in $f(x)$ with $x^2 + 4x - 2$	$fg(x) = f[g(x)] = f(x^2 + 4x - 2)$
Doing so:	$f(x^2 + 4x - 2) = 3(x^2 + 4x - 2) - 2$
Simplifying gives:	$f(x^2 + 4x - 2) = 3x^2 + 12x - 6$
The final answer is:	so $fg(x) = 3x^2 + 12x - 6$
(ii) $gf(x)$ is found by replacing all the x terms in $g(x)$ with $3x - 2$	$gf(x) = g[f(x)] = g(3x - 2)$
Doing so:	$g(3x - 2) = (3x - 2)^2 + 4(3x - 2) - 2$
Simplifying gives:	$g(3x - 2) = 9x^2 - 12x + 4 + 12x - 8 - 2 \quad = 9x^2 - 4$
The final answer is:	$so\ gf(x) = 9x^2 - 4$

It is clear from this example that $gf(x) \neq fg(x)$.

The modulus operator

The modulus of any number k, denoted as |k|, is simply its non-negative value. For example, $|-3| = 3$ and also $|3| = 3$. The modulus is sometimes called the absolute value.

Modulus functions

There are two types of modulus functions you will encounter. The first is functions of the form $y = |f(x)|$, and the second is functions of the form $y = f(|x|)$. Let's discuss each of these, starting with the former.

$y = |f(x)|$ e.g $y = |5x - 4|$

With functions of this form, we can see that the modulus is placed around the entire function. This means that the function will never output a negative number. Graphically, this means that the function will always be above (or on) the x-axis.

$y = f(|x|)$ e.g $y = \sin|x|$

Here, the modulus is applied to the inputs of the function, rather than the outputs. This means that for any number a, we have that $f(a) = f(-a)$. Graphically, this means that the function is symmetric about the y-axis.

The above interpretations lead us onto the following algorithms for sketching modulus functions of each form:

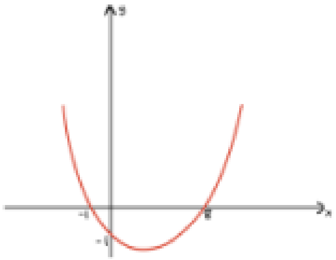
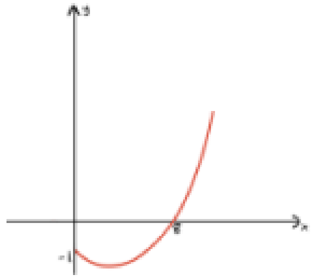
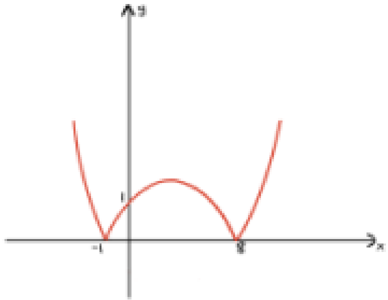
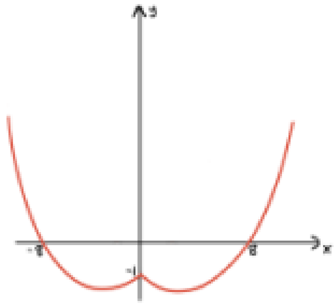
[A] $y = |f(x)|$

1) Start by sketching the graph $y = f(x)$.
2) Take the section of the graph below the x-axis and reflect it in the x-axis.

[B] $y = f(|x|)$

1) Start by sketching the function for $x \geq 0$.
2) Reflect this in the y-axis.

<u>Example 2:</u> Given that $f(x) = x^2 - 7x - 8$, sketch: (i) $|f(x)|$, (ii) $f(|x|)$

(i) $\lvert f(x)\rvert$		(ii) $f(\lvert x\rvert)$	
Step	Corresponding graph	Step	Corresponding graph
We start by sketching $y = x^2 - 7x - 8$.		Start by sketching $y = x^2 - 7x - 8$ for $x \geq 0$.	
Next, we reflect the portion of the graph below the x-axis in the x-axis.		Now reflect the graph in the y-axis.	

Inverse functions

The inverse of a function $f(x)$, denoted by $f^{-1}(x)$, is another function that "reverses" $f(x)$. It maps elements in the range of $f(x)$ back into the domain of $f(x)$. This is why inverse functions are only defined for one-to-one functions.

Here are four key points regarding inverse functions that follow from our above definition:

- An inverse function is simply the original function reflected in the line $y = x$.
- For any function $f(x)$ and its inverse $f^{-1}(x)$, we have that $ff^{-1}(x) = f^{-1}f(x) = x$.
- The domain of $f(x)$ is the range of $f^{-1}(x)$.
- The domain of $f^{-1}(x)$ is the range of $f(x)$.

You need to know how to find the inverse of a given function, $y = f(x)$. The procedure can be summarised in 3 steps:

[1] Interchange the x and y variables in your function.

[2] Rearrange your equation to make y the subject again.

[3] This new function you have found is the inverse.

Example 3: Find the inverse of the function $f(x) = x^3 - 8 \ \{x \in \mathbb{R}, x \geq 2\}$, stating its range.

[1] We have $y = x^3 - 8$, so interchanging x and y:	$x = y^3 - 8$
[2] Rearranging to make y the subject	$y = \sqrt[3]{x+8}$
[3] This function is our inverse. The range will be the same as the domain of the original function	$f^{-1}(x) = \sqrt[3]{x+8}$ Range: $y \geq 2$

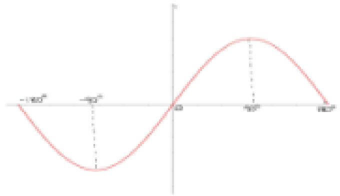
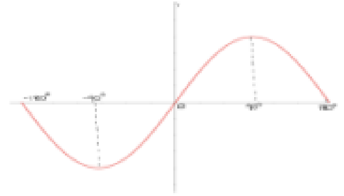
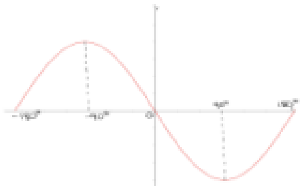
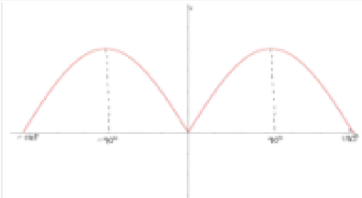
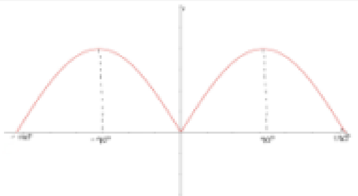
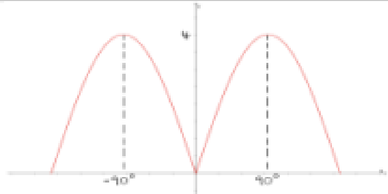
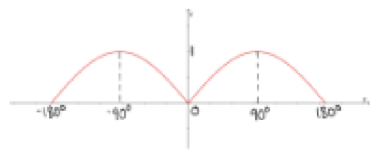
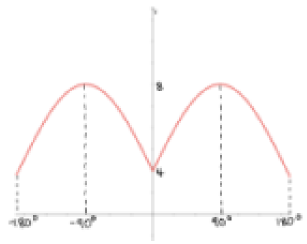
Combining Transformations

You may also be asked to sketch transformations of modulus functions. You can apply the graph transformation techniques you learnt in Chapter 4, Pure Year 1 to these problems. Recall that:

- The transformation $f(x) + a$ is a translation of $f(x)$ by 'a' units in the positive y-direction.
- The transformation $f(x + a)$ is a translation of $f(x)$ by 'a' units in the negative x-direction.
- The transformation $af(x)$ is a stretch of $f(x)$ by scale factor a in the y-direction.
- The transformation $f(ax)$ is a stretch of $f(x)$ by scale factor $\frac{1}{a}$ in the x-direction.
- The transformation $-f(x)$ is a reflection of $y = f(x)$ In the x-axis.
- The transformation $f(-x)$ is a reflection of $y = f(x)$ in the y-axis.

Let's look at how we can apply the above rules to a question involving a modulus function.

<u>Example 4</u>: Given that $f(x) = 2sinx, -180° \leq x \leq 180°$,
sketch the graphs of: (i) $\frac{1}{2}|f(-x)|$, (ii) $2f(|x|) + 4$

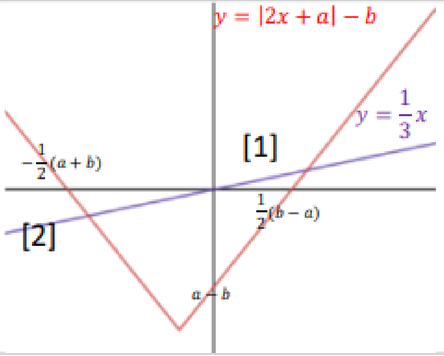
(i) ½$\|f(-x)\|$		(ii) $2f(\|x\|) + 4$	
Step	Corresponding graph	Step	Corresponding graph
We start by sketching $f(x)$		We start by sketching $f(x)$	
Now sketching $f(-x)$ (reflect the graph in the y-axis).		Now we sketch $f(\|x\|)$ by reflecting the graph of $f(x)$ for $x \geq 0$ in the y-axis.	
Applying the modulus function now to sketch $\| f(x)\|$. The portion of the graph below the x-axis is reflected in the x-axis.		Stretching the graph by a scale factor of 2 in the y-direction to achieve $2 f(\|x\|)$.	
Stretching our graph by a scale factor of ½ to attain the graph of ½$\|f(-x)\|$		Finally, we translate our graph up by 4 units giving us the graph of $2f(\|x\|) + 4$.	

Solving modulus equations

You can also be expected to be able to solve equations involving modulus and non-modulus functions. It is best to first sketch the functions within your equation so that you can figure out graphically where the solutions are located, and then use your knowledge of modulus functions to completely solve the equation.

<u>Example 5:</u> Given that a and b are constants, and that $0 < a < b$,

Solve, for x, the equation $|2x + a| - b = \frac{1}{3}x$

We start by sketching $\|2x + a\| - b$ and $y = \frac{1}{3}x$ on the same axes. Use your knowledge of combinations of transformations and modulus functions to do this. We look at our sketch and identify the points of intersection (labelled [1] and [2]). These are the points we want to find.	
Point [1] is an intersection of the non-reflected portion of $\|2x + a\| - b$ with $\frac{1}{3}x$. Therefore, if we wish to find this point, we need to solve $2x + a - b = \frac{1}{3}x$.	$2x - \frac{1}{3}x = b - a$ $\therefore \frac{5}{3}x = b - a$ so $\boldsymbol{x = \frac{3}{5}(b - a)}$ **[1]**
Point [2] is an intersection of the reflected portion of $\|2x + a\| - b$ with $\frac{1}{3}x$. Therefore, to find [2], we must solve the equation $-(2x + a) - b = \frac{1}{3}x$	$-2x - a - b = \frac{1}{3}x$ $-\frac{7}{3}x = a + b \therefore \boldsymbol{x = -\frac{3}{7}(a + b)}$ **[2]**

This process can be applied to any question where you need to solve an equation involving modulus functions. It is crucial you are able to sketch the graphs correctly. If you have a graphical calculator then you should always use it to check that your sketch is correct.