Algebraic methods Cheat Sheet

Important definitions

- A negation of a given statement is another statement that can be used to imply the given statement is incorrect.

- A contradiction is an incompatibility between two statements. In other words, the two statements cannot both be true.

- An improper fraction is one where the degree of the numerator is greater than or equal to the degree on the denominator. An example is $\frac{6x^2+5x}{x(x-1)}$, since the degree of the numerator (2) is equal to the degree of the denominator. Recall that the degree of the numerator/denominator is the highest power of x present.

Proof by contradiction

Proof by contradiction is a powerful method used to prove statements and is applicable in many mathematical contexts. The idea is relatively simple:

- We start by assuming the given statement is false.
- We work to show that this assumption leads to a contradiction, either in the assumption we made or in a fact we know to be true.

Here are some helpful facts to remember when proving statements by contradiction:

- Any even number, n, can be written in the form $n = 2k$, for some integer k.
- Any odd number, n, can be written in the form $n = 2k + 1$, for some integer k.
- Rational numbers can be written in the form $\frac{a}{b}$, where a and b are integers.
- Irrational numbers cannot be written in the form $\frac{a}{b}$.

We will now go through two key examples:

Example 1: Prove by contradiction that there are infinitely many prime numbers.

Assume there are a finite number of prime numbers.

Let's say there are n prime numbers $p_1, p_2, p_3, \dots, p_n$. Now, consider the number $K = p_1 p_2 p_3 \dots p_n + 1$. This new number leaves a remainder of 1 upon division by any of the prime numbers. This means that K is not divisible by any of the prime numbers, which in turn implies that either K is prime, or K has a prime factor that is not listed! Either way, this is a contradiction in the assumption we took to be true. Therefore, there must be an infinite number of prime numbers.

<u>Example 2:</u> Prove by contradiction that there exist no integers a and b such that $21a + 14b = 1$.

Assume there are integers a and b such that $21a + 14b = 1$.

Dividing through by 7: $3a + 2b = \frac{1}{7}$.
We can already see a problem; we assumed a and b to be integers, so $3a + 2b$ must also be an integer. As a result, there is no possible way we could have $\frac{1}{7}$.on the RHS. This is a contradiction and so we can conclude there are no integers a and b such that that $21a + 14b = 1$. Note that we choose to divide by 7 as it is a common divisor of 21 and 14.

Algebraic fractions

You need to be able to manipulate algebraic fractions in the same way as numeric fractions. Here are three short examples:

$$\frac{x}{x+4}+\frac{4}{x-1}=\frac{x(x-1)}{(x+4)(x-1)}+\frac{4(x+4)}{(x+4)(x-1)}=\frac{x(x-1)+4(x+4)}{(x+4)(x-1)}=\frac{x^2+3x+16}{(x+4)(x-1)}$$ **(Addition)**

$$\frac{x}{x+4}\times\frac{4}{x-1}=\frac{4x}{(x-1)(x+4)}$$ **(Multiplication)**

$$\frac{x}{x+4}\div\frac{4}{x-1}=\frac{x}{x+4}\times\frac{x-1}{4}=\frac{x(x-1)}{4(x+4)}$$ **(Division)**

Algebraic division

In Chapter 7 of Pure Year 1, you learnt how to divide two polynomials. We will now look at how we can rewrite an improper fraction in terms of a proper fraction using algebraic division.

If we have a fraction of the form $\frac{F(x)}{G(x)}$, where $F(x)$ and $G(x)$ are polynomials, then we can say:

(*) ▪ $\frac{F(x)}{G(x)} = Q(x) + \frac{r}{G(x)}$, where $Q(x)$ is the quotient and r is the remainder of $F(x)$ divided by $G(x)$.

We can see why this is true by looking at a non-algebraic example to begin with. Take the fraction $\frac{9}{2}$; we can express $\frac{9}{2}$ as $4 + \frac{1}{2}$, which is in the same form as the RHS of the above relationship. Now let's consider the algebraic fraction $\frac{x^3+x^2-7}{x-3}$. Using long division:

$x^2 + 4x + 12$ — This is known as the quotient, $Q(x)$

$G(x)$: $x - 3 \,\overline{)\, x^3 + x^2 + 0x - 7}$ — $F(x)$

$x^3 - 3x^2$

$4x^2 + 0x$

$4x^2 - 12x$

$12x - 7$

$12x - 36$

29 — Remainder, r

$\therefore$ we can say that $\frac{x^3+x^2-7}{x-3} = x^2 + 4x + 12 + \frac{29}{x-3}$ using (*). This new expression has no improper fractions, so we have achieved our goal.

Partial fractions

A fraction with more than one linear factor in the denominator can be split up into separate fractions, which are known as partial fractions. For example, we can rewrite $\frac{6x^2+5x-2}{x(x-1)(2x+1)}$ as $\frac{A}{x}+\frac{B}{x-1}+\frac{C}{2x+1}$ for some constants A, B and C. Notice how the linear factors in the denominator of the original fraction are now separated into different fractions.

- If you have an improper fraction, you must first perform long division and use the relationship (*) to attain an expression in terms of a proper fraction, before you can use partial fractions.

- If you have a proper fraction, you can proceed to the partial fraction method straight away.

When we say linear factor, we mean something of the form $ax + b$. Sometimes the denominator is not given in a linear factorised form. In such cases, you should try to find a factorisation if you want to split via partial fractions. For example:

$$\frac{2}{x^2-4} \to \frac{2}{(x+2)(x-2)} \quad or \quad \frac{2x}{x^2+9x+18} \to \frac{2x}{(x+6)(x+3)}$$

With the help of an example, we will go through each step of the partial fraction method.

Example 3: Split up $\frac{6x^2+5x-2}{x(x-1)(2x+1)}$ using partial fractions.

[1] We start by letting $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A}{x} + \frac{B}{x-1} + \frac{C}{2x+1}$.

[2] Next, we manipulate the RHS to make all the denominators the same:

$$\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{A(x-1)(2x+1)}{x(x-1)(2x+1)} + \frac{B(x)(2x+1)}{x(x-1)(2x+1)} + \frac{C(x)(x-1)}{x(x-1)(2x+1)}$$

[3] Now, we can equate the numerators:

$$6x^2 + 5x - 2 = A(x-1)(2x+1) + B(x)(2x+1) + C(x)(x-1)$$

We now try to find the constants A, B and C. To do so, we will use the substitution method.

We can substitute $x = 1, x = 0$ into our equation. We choose these values of x because this will result in cancellation of terms on the RHS.

Substituting $x = 1$: $\Rightarrow 6(1) + 5(1) - 2 = A(0) + B(1)(3) + C(0)$
$\Rightarrow 3B = 9 \therefore B = 3$
Substituting $x = 0$: $\Rightarrow 6(0) + 5(0) - 2 = A(-1)(1) + B(0) + C(0)$
$\Rightarrow -2 = -A \therefore A = 2$

We now know A and B so all we need to do to find C is substitute **any other** value of x into our equation.
Substituting $x = 2$: $\Rightarrow 6(4) + 5(2) - 2 = A(1)(5) + B(2)(5) + C(2)(1)$
$\Rightarrow 32 = 10 + 30 + 2C \therefore C = -4$

So, we can conclude that $\frac{6x^2+5x-2}{x(x-1)(2x+1)} \equiv \frac{2}{x} + \frac{3}{x-1} + \frac{-4}{2x+1}$, and we are done.

Repeated linear factors

Whenever there is a repeated linear factor in the denominator and we wish to use partial fractions, we have to make a slight modification to our method.

Take, for example, $\frac{2x^2+2x-18}{x(x-3)^2}$. The factor $(x-3)$ is repeated in the denominator. When we split this fraction up, we must use an extra fraction to account for the repetition. Our partial fractions become:

$$\frac{2x^2+2x-18}{x(x-3)^2} \equiv \frac{A}{x} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$$

Once you have set up the above equality, you can proceed to using partial fractions as we did above. Here are two more examples of how we split up such fractions:

$$\frac{10x^2-10x+17}{(2x+1)(x-3)^2} \equiv \frac{A}{2x+1} + \frac{B}{(x-3)} + \frac{C}{(x-3)^2}$$

$$\frac{2x}{(x+2)^2} \equiv \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$