Friday Afternoon Quiz

Integration 1

Use partial fractions followed by integration by parts

$$\int_0^\infty \left[\frac{x^2+3x+3}{(x+1)^3}\right] e^{-x} \sin x \; dx$$

Integration 1 Answer

- Start by partial fractions first

$$\frac{x^2+3x+3}{(x+1)^3} \equiv \frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$x^2+3x+3 \equiv A(x+1)^2+B(x+1)+C$$

$$x^2+3x+3 \equiv Ax^2+2Ax+A+Bx+B+C$$

$$x^2+3x+3 \equiv Ax^2+(2A+B)x+(A+B+C)$$

Hence $A=B=C=1$

- Next we find the integral of $e^{-x}\sin x$, by parts twice

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \int e^{-x}\cos x\,dx$$

e^{-x}	$-e^{-x}$
$-\cos x$	$\sin x$

e^{-x}	$-e^{-x}$
$\sin x$	$\cos x$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - \left[e^{-x}\sin x + \int e^{-x}\sin x\,dx\right]$$

$$\int e^{-x}\sin x\,dx = -e^{-x}\cos x - e^{-x}\sin x - \int e^{-x}\sin x\,dx$$

$$2\int e^{-x}\sin x\,dx = -e^{-x}(\cos x+\sin x)+C$$

$$\int e^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x+\sin x)+C$$

- Next split the integral into 3 & carry out integration by parts in the first & third integral but not in the second (left for cancelling)

$$\int_0^\infty \frac{x^2+3x+3}{(x+1)^3}\left[e^{-x}\sin x\right]dx = \int_0^\infty \frac{e^{-x}\sin x}{x+1}\,dx + \int_0^\infty \frac{e^{-x}\sin x}{(x+1)^2}\,dx + \int_0^\infty \frac{e^{-x}\sin x}{(x+1)^3}\,dx$$

$\frac{1}{x+1}$	$-\frac{1}{(x+1)^2}$
$-\frac{1}{2}e^{-x}(\cos x+\sin x)$	$e^{-x}\sin x$

$e^{-x}\sin x$	$-e^{-x}\sin x + e^{-x}\cos x = e^{-x}(\cos x-\sin x)$
$-\frac{1}{2(x+1)^2}$	$\frac{1}{(x+3)^3}$

$$\int_0^\infty \frac{x^2+3x+3}{(x+1)^3}\left[e^{-x}\sin x\right]dx = \left[-\frac{1}{2}\frac{e^{-x}}{x+1}(\sin x+\cos x)\right]_0^\infty - \int_0^\infty \frac{e^{-x}(\sin x+\cos x)}{2(x+1)^2}\,dx$$

$$+\int_0^\infty \frac{e^{-x}\sin x}{(x+1)^2}\,dx$$

$$\left[-\frac{1}{2}\frac{e^{-x}\sin x}{(x+1)^2}\right]_0^\infty + \int_0^\infty \frac{e^{-x}(\cos x-\sin x)}{2(x+1)^2}\,dx$$

$$= \left[0-\left(-\frac{1}{2}\right)\right] + \int_0^\infty -\frac{1}{2}\frac{e^{-x}\sin x}{(x+1)^2} - \frac{1}{2}\frac{e^{-x}\cos x}{(x+1)^2}\,dx$$

$$+\int_0^\infty \frac{e^{-x}\sin x}{(x+1)^2}\,dx$$

$$\left[0-(0)\right] + \int_0^\infty \frac{1}{2}\frac{e^{-x}\cos x}{(x+1)^2} - \frac{1}{2}\frac{e^{-x}\sin x}{(x+1)^2}\,dx$$

$$= \frac{1}{2}$$

Integration 2

By suitably rewriting the numerator of the integrand, find a simplified expression for the following integral.

$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x}\, dx\,.$$

Integration 2 Answer

$$\frac{12}{13}x+\frac{5}{13}\ln\left|2\cos x+3\sin x\right|+C$$

$$\int \frac{3\cos x + 2\sin x}{2\cos x + 3\sin x}\,dx = ?$$

- MANIPULATE AS FOLLOWS

$$\frac{d}{dx}\left[2\cos x + 3\sin x\right] = -2\sin x + 3\cos x$$

- REWRITE THE NUMERATOR AS

$$3\cos x + 2\sin x \equiv A\left(2\cos x + 3\sin x\right) + B\left(3\cos x - 2\sin x\right)$$

SO IT CAN BE DIVIDED BY THE DENOMINATOR

SO IT BECOMES OF THE FORM $\int \frac{f'(x)}{f(x)}\,dx$

- HENCE

$$\left.\begin{array}{l} 2A + 3B = 3 \\ 3A - 2B = 2 \end{array}\right\} \begin{array}{l}\times 2 \\ \times 3\end{array} \Rightarrow \left.\begin{array}{l} 4A + 6B = 6 \\ 9A - 6B = 6 \end{array}\right\} \Rightarrow 13A = 12$$

$$\boxed{A = \frac{12}{13}}$$

$$\& \quad 2\left(\frac{12}{13}\right) + 3B = 3$$

$$24 + 39B = 39$$

$$39B = 15$$

$$\boxed{B = \frac{5}{13}}$$

- RETURNING TO THE INTEGRAL

$$\dots = \int \frac{12}{13}\left(\frac{2\cos x + 3\sin x}{2\cos x + 3\sin x}\right) + \frac{5}{13}\left(\frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\right)\,dx$$

$$= \int \frac{12}{13} + \frac{5}{13}\left(\frac{3\cos x - 2\sin x}{2\cos x + 3\sin x}\right)\,dx$$

$$= \frac{12}{13}x + \frac{5}{13}\ln\left|2\cos x + 3\sin x\right| + C$$

Integration 3

Use trigonometric identities to find a simplified expression for

$$\int \frac{\sin^8 x - \cos^8 x}{1 - \frac{1}{2}\sin^2 2x}\, dx\ .$$

Integration 3 Answer

$$-\frac{1}{2}\sin 2x + C$$

$$\int \frac{\sin^8 x - \cos^8 x}{1-\frac{1}{2}\sin^2 2x}\,dx$$

① STARTING FROM THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE SINE DOUBLE ANGLE IN THE DENOMINATOR

$$\dots = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x+\cos^4 x)}{1-\frac{1}{2}(2\sin x\cos x)^2}\,dx = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{1-2\sin^2 x\cos^2 x}\,dx$$

② NEXT CREATE A PERFECT SQUARE IN THE DENOMINATOR AS FOLLOWS

$$\dots = \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x+\cos^4 x)}{1^2-2\sin^2 x\cos^2 x}\,dx = \int \frac{(\sin^4 x-\cos^4 x)(\sin^4 x+\cos^4 x)}{(\sin^2 x+\cos^2 x)^2-2\sin^2 x\cos^2 x}\,dx$$

③ EXPAND THE DIFFERENCE OF SQUARES IN THE NUMERATOR & THE BRACKET IN THE DENOMINATOR

$$\dots = \int \frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)(\sin^4 x+\cos^4 x)}{\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x-2\sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{-\cos 2x(\sin^4 x+\cos^4 x)}{\cos^4 x+\sin^4 x}\,dx = \int -\cos 2x\,dx$$

$$= -\frac{1}{2}\sin 2x + C$$

Decision Maths

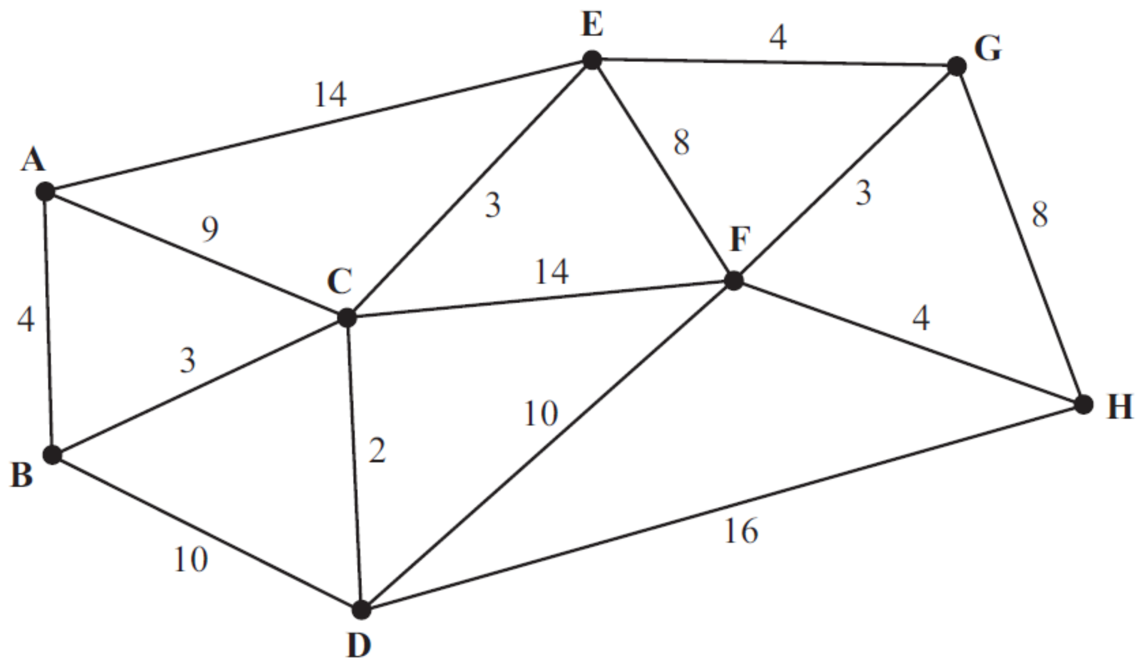


Figure 1

Figure 1 shows a network of roads between eight villages, A, B, C, D, E, F, G and H. The number on each arc gives the length, in miles, of the corresponding road.

(a) Use Dijkstra's algorithm to find the shortest distance from A to H.

Decision Maths Answer

The length of the shortest route is 21 miles

Shortest route: A B C E G F H

Name the artist

According to one American magazine, they produced some of the best albums in 2019

Name the artist Answers

James Blake

Joe Armon Jones

King Princess

Bon Iver

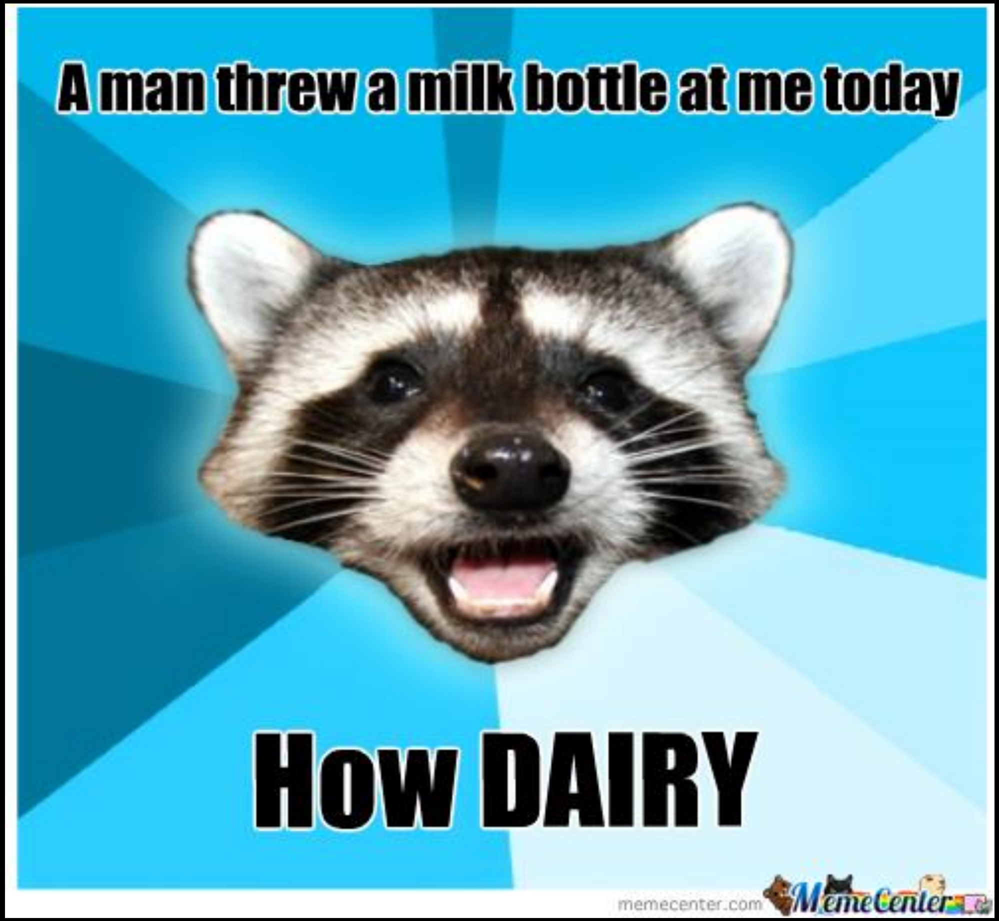
Billy Eilish

According to one American magazine, they produced some of the best albums in 2019

Name the chocolate bar

Name the chocolate bar Answers

Is this funny?



Is this funny? YES