Pure Test 3. 100 marks. 2 hours

1. The curve C has parametric equations

$$x = 3t - 4, \quad y = 5 - \frac{6}{t}, \quad t > 0$$

(a) Find $\frac{dy}{dx}$ in terms of t

(2)

The point P lies on C where $t = \frac{1}{2}$

(b) Find the equation of the tangent to C at the point P. Give your answer in the form $y = px + q$, where p and q are integers to be determined.

(3)

(c) Show that the cartesian equation for C can be written in the form

$$y = \frac{ax+b}{x+4}, \qquad x > 4$$

where a and b are integers to be determined.

(3)

(Total 8 marks)

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2. $$f(x) = (2 + kx)^{-3}, \qquad |kx| < 2, \text{ where } k \text{ is a positive constant}$$

The binomial expansion of f(x), in ascending powers of x, up to and including the term in x^2 is

$$A + Bx + \frac{243}{16}x^2$$

where A and B are constants.

(a) Write down the value of A.

(1)

(b) Find the value of k.

(3)

(c) Find the value of B.

(2)

(Total 6 marks)

3.

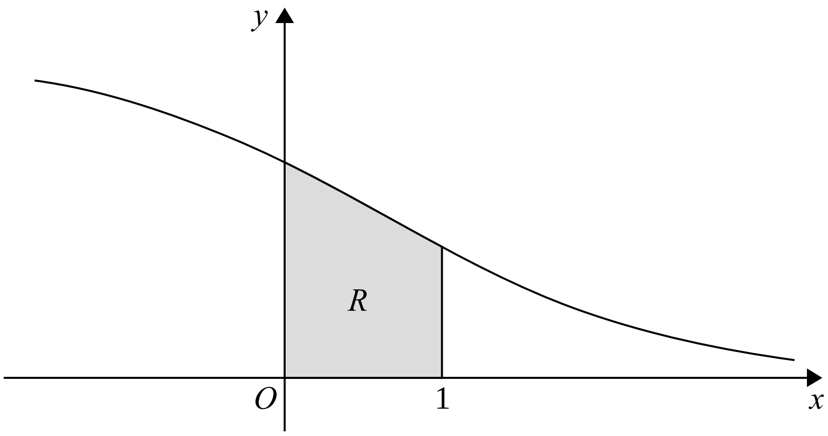

Figure 1

Figure 1 shows a sketch of part of the curve with equation $y = \dfrac{6}{(e^x + 2)}, \quad x \in \mathbb{R}$

The finite region R, shown shaded in Figure 1, is bounded by the curve, the y-axis, the x-axis and the line with equation $x = 1$

The table below shows corresponding values of x and y for $y = \dfrac{6}{(e^x + 2)}$

x	0	0.2	0.4	0.6	0.8	1
y	2		1.71830	1.56981	1.41994	1.27165

(*a*) Complete the table above by giving the missing value of y to 5 decimal places.

(1)

(*b*) Use the trapezium rule, with all the values of y in the completed table, to find an estimate for the area of R, giving your answer to 4 decimal places.

(3)

(*c*) Use the substitution $u = e^x$ to show that the area of R can be given by

$$\int_a^b \frac{6}{u(u+2)}\,du$$

where a and b are constants to be determined.

(2)

(*d*) Hence use calculus to find the exact area of R.
[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

(Total 12 marks)

4. The curve C has equation

$$4x^2 - y^3 - 4xy + 2^y = 0$$

The point P with coordinates $(-2, 4)$ lies on C.

(*a*) Find the exact value of $\frac{dy}{dx}$ at the point P.

(6)

The normal to C at P meets the y-axis at the point A.

(*b*) Find the y coordinate of A, giving your answer in the form $p + q\ln 2$, where p and q are constants to be determined.

(3)

(Total 9 marks)

5.

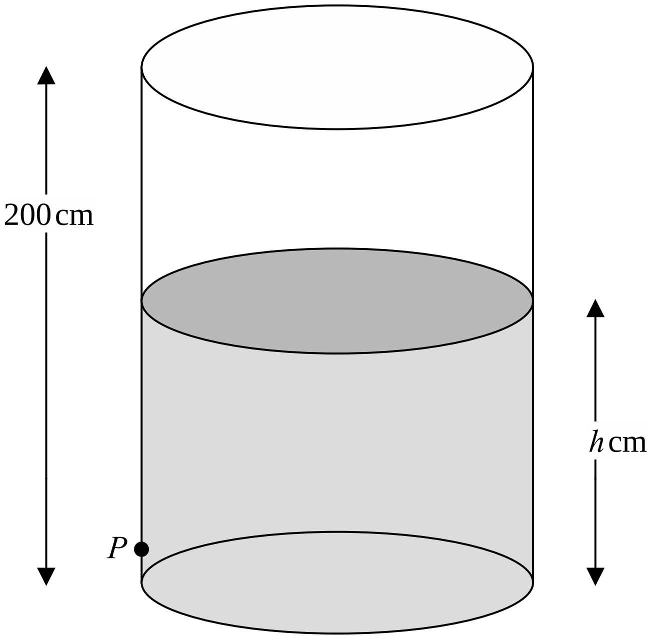

Figure 3

Figure 3 shows a vertical cylindrical tank of height 200 cm containing water.
Water is leaking from a hole P on the side of the tank.

At time t minutes after the leaking starts, the height of water in the tank is h cm.

The height h cm of the water in the tank satisfies the differential equation

$$\frac{dh}{dt} = k(h-9)^{\frac{1}{2}}, \qquad 9 < h \leqslant 200$$

where k is a constant.

Given that, when $h = 130$, the height of the water is falling at a rate of 1.1 cm per minute,

(a) find the value of k.

(2)

Given that the tank was full of water when the leaking started,

(b) solve the differential equation with your value of k, to find the value of t when $h = 50$

(6)

(Total 8 marks)

6. Express $\frac{4x}{x^2-9} - \frac{2}{x+3}$ as a single fraction in its simplest form.

(Total 4 marks)

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7. Find the exact solutions, in their simplest form, to the equations

(a) $e^{3x-9} = 8$

(3)

(b) $\ln(2y + 5) = 2 + \ln(4 - y)$

(4)

(Total 7 marks)

8.

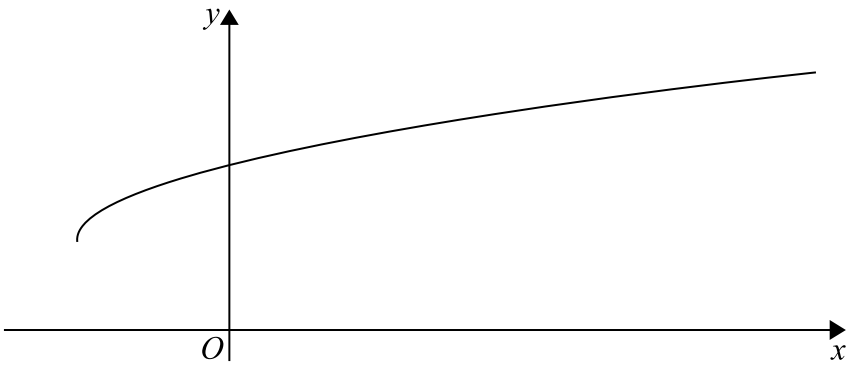

Figure 1

Figure 1 shows a sketch of part of the graph of $y = \mathrm{g}(x)$, where

$$g(x) = 3 + \sqrt{x + 2}, \qquad x \geq -2$$

(a) State the range of g.

(1)

(b) Find $\mathrm{g}^{-1}(x)$ and state its domain.

(3)

(c) Find the exact value of x for which

$$\mathrm{g}(x) = x$$

(4)

(d) Hence state the value of a for which

$$\mathrm{g}(a) = \mathrm{g}^{-1}(a)$$

(1)

(Total 9 marks)

9. (*a*) Write $5 \cos \theta - 2 \sin \theta$ in the form $R \cos (\theta + \alpha)$, where R and α are constants,

$R > 0$ and $0 \leqslant \alpha < \frac{\pi}{2}$

Give the exact value of R and give the value of α in radians to 3 decimal places.

(3)

(*b*) Show that the equation

$$5 \cot 2x - 3 \operatorname{cosec} 2x = 2$$

can be rewritten in the form

$$5 \cos 2x - 2 \sin 2x = c$$

where c is a positive constant to be determined.

(2)

(*c*) Hence or otherwise, solve, for $0 \leqslant x < \pi$,

$$5 \cot 2x - 3 \operatorname{cosec} 2x = 2$$

giving your answers to 2 decimal places.

(*Solutions based entirely on graphical or numerical methods are not acceptable*.)

(4)

(Total 9 marks)

10.

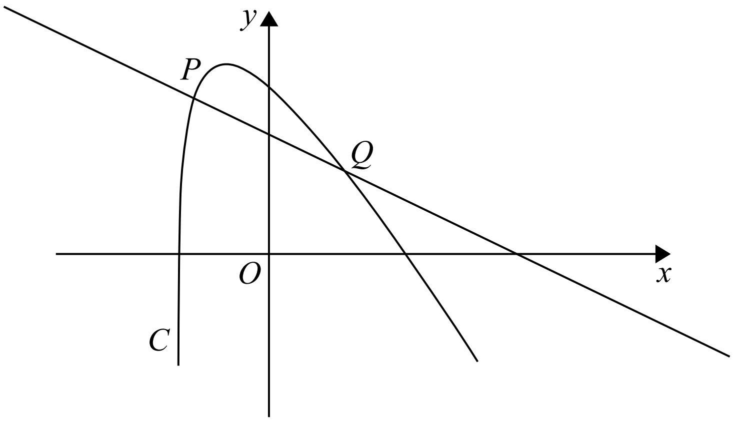

Figure 2

Figure 2 shows a sketch of part of the curve C with equation

$$y = 2\ln(2x+5) - \frac{3x}{2}, \qquad x > -2.5$$

The point P with x coordinate –2 lies on C.

(a) Find an equation of the normal to C at P. Write your answer in the form $ax + by = c$, where a, b and c are integers.

(5)

The normal to C at P cuts the curve again at the point Q, as shown in Figure 2.

(b) Show that the x coordinate of Q is a solution of the equation

$$x = \frac{20}{11}\ln(2x+5) - 2$$

(3)

The iteration formula

$$x_{n+1} = \frac{20}{11}\ln(2x_n+5) - 2$$

can be used to find an approximation for the x coordinate of Q.

(c) Taking $x_1 = 2$, find the values of x_2 and x_3, giving each answer to 4 decimal places.

(2)

(Total 10 marks)

11. Given that a and b are positive constants,

(a) on separate diagrams, sketch the graph with equation

(i) $y = |2x - a|$

(ii) $y = |2x - a| + b$

Show, on each sketch, the coordinates of each point at which the graph crosses or meets the axes.

(4)

Given that the equation

$$|2x - a| + b = \frac{3}{2}x + 8$$

has a solution at $x = 0$ and a solution at $x = c$,

(b) find c in terms of a.

(4)

(Total 8 marks)

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12. (i) Given $y = 2x(x^2 - 1)^5$, show that

(a) $\frac{dy}{dx} = g(x)(x^2 - 1)^4$ where $g(x)$ is a function to be determined.

(4)

(b) Hence find the set of values of x for which $\frac{dy}{dx} \geqslant 0$

(2)

(ii) Given

$$x = \ln(\sec 2y), \qquad 0 < y < \frac{\pi}{4}$$

find $\frac{dy}{dx}$ as a function of x in its simplest form.

(4)

(Total 10 marks)
