

Pure Test 54 marks 65 minutes

1. $f(x) = \frac{6}{x} + \frac{3}{x^2} - 7x^{\frac{5}{2}}$

(a) Find $\int f(x) dx$.

(3 marks)

(b) Evaluate $\int_4^9 f(x) dx$, giving your answer in the form $m + n \ln p$, where m, n and p are rational numbers.

(3 marks)

2. Prove by contradiction that there are infinitely many prime numbers.

(6 marks)

3. A large arch is planned for a football stadium. The parametric equations of the arch are $x = 8(t+10)$, $y = 100 - t^2$, $-19 \leq t \leq 10$ where x and y are distances in metres. Find

(a) the cartesian equation of the arch,

(3 marks)

(b) the width of the arch,

(2 marks)

(c) the greatest possible height of the arch.

(2 marks)

4. (a) Given that $f(x) = \sin x$, show that

$$f'(x) = \lim_{h \rightarrow 0} \left(\left(\frac{\cos h - 1}{h} \right) \sin x + \frac{\sin h}{h} \cos x \right)$$

(4 marks)

(b) Hence prove that $f'(x) = \cos x$.

(2 marks)

5. Figure 1 shows the right-angled triangles ΔABC , ΔABD and ΔBDC , with $AB = 1$ and $\angle BAD = \theta$.

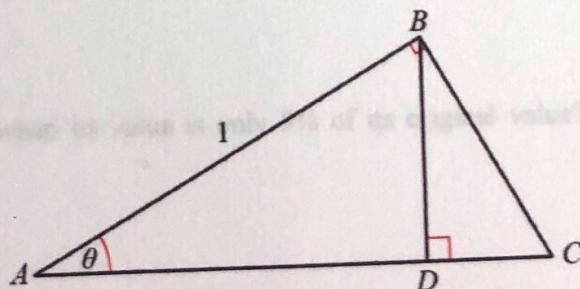


Figure 1

Prove that $1 + \tan^2 \theta = \sec^2 \theta$.

(8 marks)

6. $p(t) = \frac{1}{10} \ln(t+1) - \cos\left(\frac{t}{2}\right) + \frac{1}{10} t^{\frac{3}{2}}, \quad 0 \leq t \leq 12$

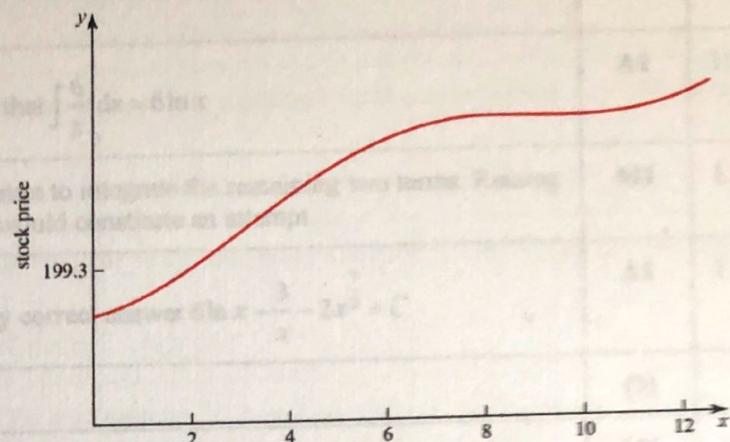


Figure 2

Figure 2 is a graph of the price of a stock during a 12-hour trading window. The equation of the curve is given above.

- (a) Show that the price reaches a local maximum in the interval $8.5 < t < 8.6$.

(5 marks)

Figure 3 shows that the price reaches a local minimum between 9 and 11 hours after trading begins.

- (b) Using the Newton-Raphson procedure once and taking $t_0 = 9.9$ as a first approximation, find a second approximation of when the price reaches a local minimum.

(6 marks)

7. The value of a computer, V , decreases over time, t , measured in years. The rate of decrease of the value is proportional to the remaining value.

Given that the initial value of the computer is V_0 ,

- (a) show that $V = V_0 e^{-kt}$.

(4 marks)

After 10 years the value of the computer is $\frac{1}{5}V_0$.

- (b) Find the exact value of k .

(3 marks)

- (c) How old is the computer when its value is only 5% of its original value? Give your answer to 3 significant figures.

(3 marks)

Pure Test

$$\textcircled{1} \text{a) } \int f(x) dx = \int \frac{6}{x} + \frac{3}{x^2} - 7x^{5/2} dx$$

$$= 6\ln x - \frac{3}{x} - 2x^{7/2} + C \quad \text{MIAI}$$

$$\text{b) } \left[6\ln x - \frac{3}{x} - 2x^{7/2} \right]_4^9 = (6\ln 9 - \frac{1}{3} - 4374) - (6\ln 4 - \frac{3}{4} - 256) \\ = 6\ln \frac{9}{4} - \frac{4941}{12} \quad (\text{or } 12\ln \frac{3}{2} - \frac{4941}{12}) \quad \text{MIAI}$$

- (2) Assume there is a finite amount of prime numbers B1
Let all the prime numbers be $p_1, p_2, p_3, \dots, p_n$ M1
- Consider $N = p_1 p_2 p_3 \dots p_n + 1$ M1
- Now, when N is divided by any prime number,
there will be a remainder of 1 M1
- $\therefore N$ is either prime or N has a prime factor which
is not one of $p_1, p_2, p_3, \dots, p_n$ B1
- ∴ contradiction
- ∴ There is an infinite number of prime numbers B1

$$\textcircled{3} \quad a) \quad \alpha = 8(t+10) \Rightarrow t = \frac{x-10}{8}$$

M1

$$y = 100 - t^2 \Rightarrow y = 100 - \left(\frac{x}{8} - 10\right)^2$$

$$\therefore y = 100 - \left(\frac{x^2}{64} - \frac{5x}{2} + 100\right)$$

$$\therefore y = -\frac{x^2}{64} + \frac{5x}{2}$$

MIAI

b) The extreme values of x occur when $y=0$

$$\therefore 100 - t^2 = 0$$

$$\left(\text{or } -\frac{x^2}{64} + \frac{5x}{2} = 0 \right)$$

$$\therefore t = \pm 10$$

$$\therefore x = 0 \text{ or } x = \frac{5 \times 64}{2} \\ = 160$$

$$\therefore x = 0 \text{ or } x = 160$$

MIAI

\therefore The width is 160

c) greatest height occurs when $\frac{dy}{dx} = 0$

$$\therefore -\frac{2x}{64} + \frac{5}{2} = 0$$

(OR greatest height is at the centre of the arch, when $x=80$)

$$\therefore x = \frac{5 \times 64}{4} = 80$$

$$\left(\text{OR } y = -\frac{80^2}{64} + \frac{5 \times 80}{2} = 100 \right)$$

$$\therefore t = 0$$

MIAI

$$\therefore y = 100$$

(4) a) $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ M1

$$= \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin(x)}{h}$$
 M1
$$= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$
 M1
$$= \lim_{h \rightarrow 0} \frac{\sin x (\cosh - 1) + \sinh x \cos x}{h}$$

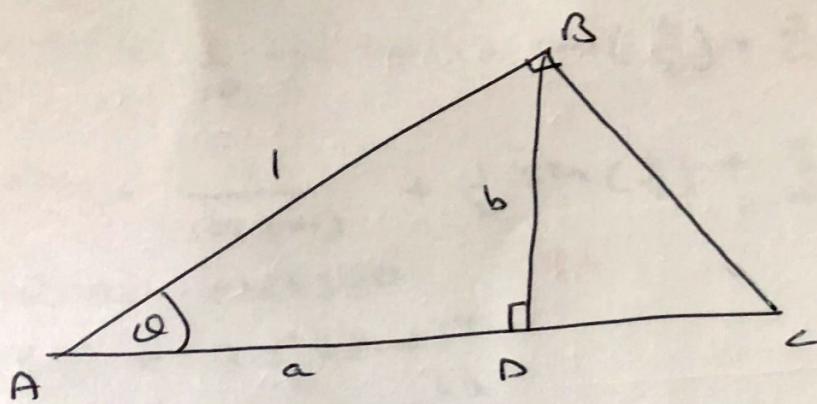
$$= \lim_{h \rightarrow 0} \left(\frac{\cosh - 1}{h} \right) \sin x + \frac{\sinh x \cos x}{h}$$
 A1

b) As $h \rightarrow 0$, $\cosh \rightarrow 1$, $\therefore \frac{\cosh - 1}{h} \rightarrow 0$

As $h \rightarrow 0$ $\sinh \rightarrow h$, $\therefore \frac{\sinh}{h} \rightarrow 1$

$$\therefore f'(x) = (0)(\sin x) + \cos x$$
 A1
$$= \cos x$$

(5)



M1

$$\tan \theta = \frac{b}{a}$$

$$\therefore 1 + \tan^2 \theta = 1 + \frac{b^2}{a^2}$$

M1

$$= \frac{a^2 + b^2}{a^2}$$

B1

$$\text{But } a^2 + b^2 = 1 \quad (\text{Pythagoras})$$

M1

$$\therefore 1 + \tan^2 \theta = \frac{1}{a^2}$$

M1

$$\text{Now, } \cos \theta = \frac{a}{l}$$

A1

$$\therefore \sec \theta = \frac{l}{a}$$

M1

$$\therefore \sec^2 \theta = \frac{l^2}{a^2}$$

A1

$$\therefore 1 + \tan^2 \theta = \sec^2 \theta$$

(6)

$$a) P(t) = \frac{1}{10} \ln(t+1) - \cos\left(\frac{t}{2}\right) + \frac{1}{10} t^{3/2}$$

$$\therefore P'(t) = \frac{1}{10(t+1)} + \frac{1}{2} \sin\left(\frac{t}{2}\right) + \frac{3}{20} t^{1/2} \quad M(1A)$$

Consider $P'(t) = 0$ B1

$$P'(8.5) = 3.53 \times 10^{-4}$$

$$P'(8.6) = -7.78 \times 10^{-3}$$

$\left. \begin{array}{l} \\ \end{array} \right\} M1$

There is a change of sign and the function is continuous.

\therefore The gradient = 0 in the interval $[8.5, 8.6]$ and so A1
there is a local maximum

b) Local minimum occurs when $P'(t) = 0$ B1

$$P''(t) = -\frac{1}{10(t+1)^2} + \frac{1}{4} \cos\left(\frac{t}{2}\right) + \frac{3}{40} t^{-1/2} \quad M(1A)$$

$$t_0 = 9.9$$

$$\therefore t_1 = 9.9 - \frac{P'(9.9)}{P''(9.9)}$$

$$= 9.9 - \frac{-0.0048}{0.0818} = 9.959 \quad (3 \text{ dp}) \quad M(1A)$$

$$7) a) \frac{dV}{dt} = -kV$$

M1

$$\therefore \int \frac{dV}{V} = -k \int dt$$

M1

$$\therefore \ln V = -kt + C$$

A1

$$\therefore V = e^{-kt+C}$$

$$\therefore V = e^{-kt} e^C$$

$$\therefore V = V_0 e^{-kt} \quad \text{where } V_0 = e^C \quad A1$$

$$b) \text{ when } t=10, V = \frac{1}{5} V_0$$

$$\therefore \frac{1}{5} V_0 = V_0 e^{-10k}$$

M1

$$\therefore \frac{1}{5} = e^{-10k}$$

$$\therefore -10k = \ln \frac{1}{5}$$

M1

$$\therefore k = -\frac{1}{10} \ln \frac{1}{5} \quad (\text{or } \frac{1}{10} \ln 5) \quad A1$$

$$c) \text{ At 5%, } V = \frac{1}{20} V_0$$

M1

$$\therefore \frac{1}{20} V_0 = V_0 e^{-5k}$$

$$\therefore e^{-5k} = \frac{1}{20}$$

$$\therefore -5k = \ln \frac{1}{20}$$

$$\therefore k = -\frac{\ln \frac{1}{20}}{5} = \frac{18.6 \text{ years}}{5} = 3.72 \text{ years} \quad M1A1$$