Answer ALL questions. Write your answers in the spaces p

1.

$$f(x) = 3x^3 + 2ax^2 - 4x + 5a$$

Given that $(x + 3)$ is a factor of $f(x)$, find the value of the constant a.

If $x+3$ is a factor, $f(-3)=0$ M1

$\therefore -81 + 18a + 12 + 5a = 0$

$\therefore 23a = 69$ M1

$a = 3$ A1

Question 2 continued

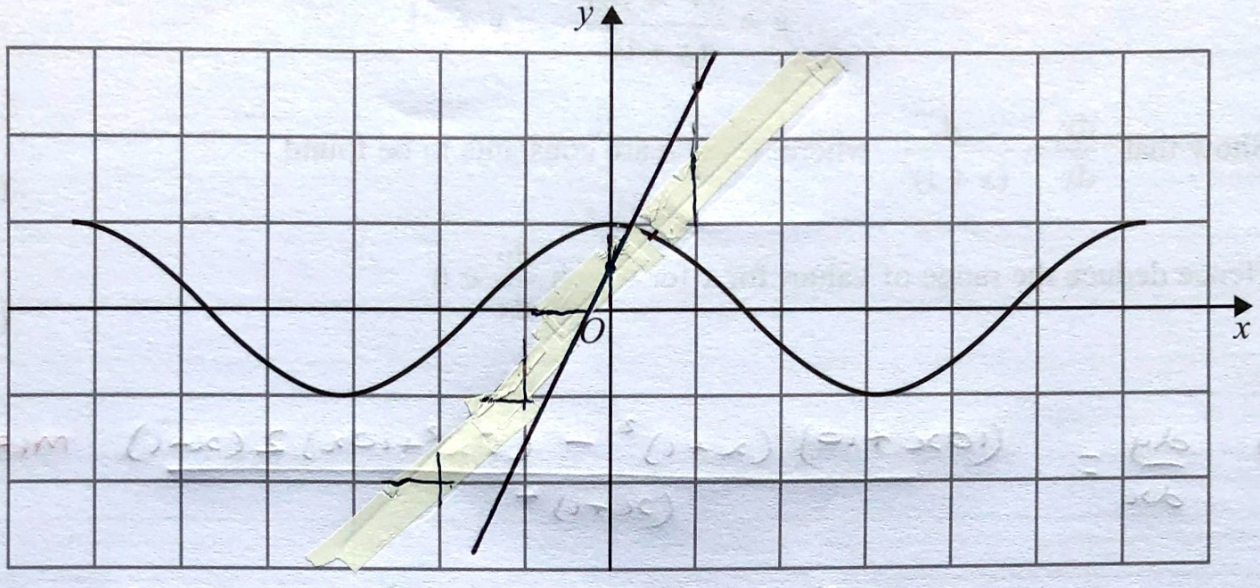


Diagram 1

a) sketch $y = 2x + \frac{1}{2}$ so graphs meet where

$$\cos 2x = 2x + \frac{1}{2}$$

$$\therefore \cos 2x - 2x - \frac{1}{2} = 0$$

It is clear from the diagram that the graphs only cross in one place.

$\therefore$ $\cos 2x - 2x - \frac{1}{2} = 0$ has only one root B1

b) For small angles $\cos x \approx 1 - \frac{x^2}{2}$ M1

$$\therefore 1 - \frac{x^2}{2} - 2x - \frac{1}{2} = 0$$

$$\therefore 2 - x^2 - 4x - 1 = 0$$

$$\therefore x^2 + 4x - 1 = 0$$ dM1

$$\therefore x = -2 + \sqrt{5}, -2 - \sqrt{5}$$

since the root is α, α is small and $\alpha > 0$, $\alpha = 0.236$ (3dp)

A1

3. $$y = \frac{5x^2 + 10x}{(x+1)^2} \qquad x \neq -1$$

(a) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{A}{(x+1)^n}$ where A and n are constants to be found.

(4)

(b) Hence deduce the range of values for x for which $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$

(1)

a) $$\frac{dy}{dx} = \frac{(10x+10)(x+1)^2 - (5x^2+10x)2(x+1)}{(x+1)^4}$$ M1A1

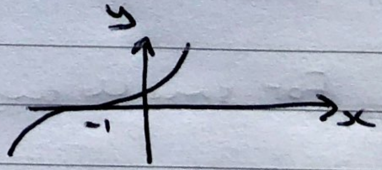
$$= \frac{(x+1)\left[(10x+10)(x+1) - 2(5x^2+10x)\right]}{(x+1)^4}$$

$$= \frac{10x^2+10x+10x+10-10x^2-20x}{(x+1)^3}$$ M1

$$= \frac{10}{(x+1)^3}$$ A1

$\therefore A = 10, \; n = 3$

b) $\frac{dy}{dx} < 0 \Rightarrow (x+1)^3 < 0$



$\therefore x < -1$ B1

4. (a) Find the first three terms, in ascending powers of x, of the binomial expansion of

$$\frac{1}{\sqrt{4-x}}$$

giving each coefficient in its simplest form.

(4)

The expansion can be used to find an approximation to $\sqrt{2}$
Possible values of x that could be substituted into this expansion are:

- $x = -14$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{18}} = \dfrac{\sqrt{2}}{6}$
- $x = 2$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$
- $x = -\dfrac{1}{2}$ because $\dfrac{1}{\sqrt{4-x}} = \dfrac{1}{\sqrt{\dfrac{9}{2}}} = \dfrac{\sqrt{2}}{3}$

(b) Without evaluating your expansion,

(i) state, giving a reason, which of the three values of x should not be used

(1)

(ii) state, giving a reason, which of the three values of x would lead to the most accurate approximation to $\sqrt{2}$

(1)

a) $\dfrac{1}{\sqrt{4-x}} = (4-x)^{-1/2}$ M1

$= 4^{-\frac{1}{2}}\left(1-\dfrac{x}{4}\right)^{-1/2}$

M1 A1

$= \dfrac{1}{2}\left(1 + (-\tfrac{1}{2})(-\tfrac{x}{4}) + \dfrac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\left(\dfrac{-x}{4}\right)^2 + \dots\right)$

$= \dfrac{1}{2}\left(1 + \dfrac{x}{8} + \dfrac{3x^2}{128}\right) = \dfrac{1}{2} + \dfrac{x}{16} + \dfrac{3x^2}{256}$ A1

b) valid for $\left|-\dfrac{x}{4}\right| < 1$ i.e. $\left|\dfrac{x}{4}\right| < 1$ i.e. $|x| < 4$

i) if $|x| < 4$, $x \neq -14$ B1

ii) the smaller modulus value of x will be most accurate

so $x = -\frac{1}{2}$ B1

5.

$$f(x) = 2x^2 + 4x + 9 \qquad x \in \mathbb{R}$$

(a) Write $f(x)$ in the form $a(x+b)^2 + c$, where a, b and c are integers to be found.

(3)

(b) Sketch the curve with equation $y = f(x)$ showing any points of intersection with the coordinate axes and the coordinates of any turning point.

(3)

(c) (i) Describe fully the transformation that maps the curve with equation $y = f(x)$ onto the curve with equation $y = g(x)$ where

$$g(x) = 2(x-2)^2 + 4x - 3 \qquad x \in \mathbb{R}$$

(ii) Find the range of the function

$$h(x) = \frac{21}{2x^2 + 4x + 9} \qquad x \in \mathbb{R}$$

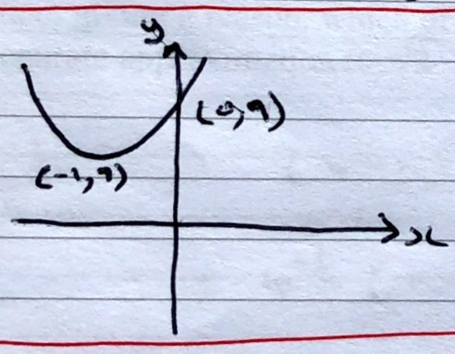
(4)

a) $f(x) = \overset{B1}{2}\left(x^2 + 2x + \frac{9}{2}\right)$

$= 2\left[(x+1)^2 + \frac{7}{2}\right]$

$= 2(x+1)^2 + 7$ M1A1 (3)

b)



U (not through (0,0)) B1

(0,9) B1

min (-1,7) B1 ft (3)

c) i) $g(x) = 2(x-2)^2 + 4x - 3$

$= 2(x-2)^2 + 4(x-2) + 5$

$\therefore$ transformation is a translation $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ M1 A1

ii) Min value of $2x^2 + 4x + 9$ is 7 $\quad \frac{21}{7}$ M1

Max value of $2x^2 + 4x + 9 \to \infty$

$\therefore$ range of $h(x)$ is $0 < h(x) \leq 3$ A1

ft from (a) (4)

6. (a) Solve, for $-180° \leqslant \theta \leqslant 180°$, the equation

$$5\sin 2\theta = 9\tan\theta$$

giving your answers, where necessary, to one decimal place.

[*Solutions based entirely on graphical or numerical methods are not acceptable.*]

(6)

(b) Deduce the smallest positive solution to the equation

$$5\sin(2x - 50°) = 9\tan(x - 25°)$$

(2)

a) $5\sin 2\theta = 9\tan\theta$

$\therefore 10\sin\theta\cos\theta = \dfrac{9\sin\theta}{\cos\theta}$ M1

$\therefore 10\sin\theta\cos^2\theta = 9\sin\theta$ A1

$\therefore \sin\theta = 0$ or $\cos^2\theta = 9/10 \Rightarrow \cos\theta = \dfrac{\pm 3}{\sqrt{10}}$

B1 M1A1 A1 6

$\therefore \theta = -180°, 0, 180°, 18.4°, -18.4°, 161.6, -161.6$

M1

b) $5\sin(2x - 50°) = 9\tan(x - 25°)$

$\Rightarrow 5\sin(2(x-25)) = 9\tan x - 25$

This is the same equation as (a) with x replaced by $x - 25$

$\therefore$ smallest positive solution is where $x - 25 = -18.4°$

$\therefore x = 6.6$ A1 2

7. In a simple model, the value, £V, of a car depends on its age, t, in years.

The following information is available for car A

- its value when new is £20 000
- its value after one year is £16 000

(a) Use an exponential model to form, for car A, a possible equation linking V with t.

(4)

The value of car A is monitored over a 10-year period.
Its value after 10 years is £2 000

(b) Evaluate the reliability of your model in light of this information.

(2)

The following information is available for car B

- it has the same value, when new, as car A
- its value depreciates more slowly than that of car A

(c) Explain how you would adapt the equation found in (a) so that it could be used to model the value of car B.

(1)

a) $V = Ae^{kt}$ M1

$t=0 \Rightarrow V = 20000 \quad \therefore \; 20000 = A$ M1

$t=1 \Rightarrow V = 16000 \quad \therefore \; 16000 = 20000e^{k}$

$\therefore \; e^{k} = 0.8$

$\therefore \; k = \ln 0.8 = -0.223$ M1A1 4

$\therefore \; V = 20000e^{-0.223t}$

b) $t=10 \Rightarrow V = 2000$

use $V = 20000e^{-0.223t}$ M1

$t=10 \Rightarrow V = 20000e^{-2.23} \approx 2150$

$\therefore$ The model is fairly reliable as $2150 \simeq 2000$ A1 2

c) increase k, e.g. $V = 20000e^{-0.2t}$ B1 1

Alternative a) $V = 20000 \times 0.8^{t}$

Question 8 continued

a) The graph cuts the x-axis at $(-2, 0)$, $(4, 0)$ and $(0, 0)$

$$\therefore\ R_1 = \int_{-2}^{0} x(x+2)(x-4)\,dx$$

$$= \int_{-2}^{0} x(x^2-2x-8)\,dx$$

$$= \int_{-2}^{0} x^3 - 2x^2 - 8x\,dx \qquad \text{B1}$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - 4x^2\right]_{-2}^{0} \qquad \text{M1A1}$$

$$= 0 - \left(4 + \tfrac{16}{3} - 16\right) = \tfrac{20}{3} \qquad \text{A1}$$

(4)

b) $$R_2 = \int_0^b x^3 - 2x^2 - 8x\,dx$$

$$= \left[\tfrac{1}{4}x^4 - \tfrac{2}{3}x^3 - 4x^2\right]_0^b$$

$$= \frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2$$

but $R_2 = R_1$

$$\therefore\ \frac{b^4}{4} - \frac{2}{3}b^3 - 4b^2 = -\frac{20}{3} \qquad \text{M1}$$ since R_2 is under x-axis

$$\therefore\ 3b^4 - 8b^3 - b^2 = -80 \qquad \text{A1}$$

$$\therefore\ 3b^4 - 8b^3 - 48b^2 + 80 = 0$$

Now $(b+2)^2(3b^2-20b+20)$

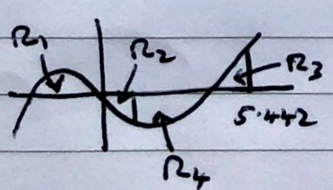
$$= (b^2+4b+4)(3b^2-20b+20) \qquad \text{M1}$$

$$= 3b^4 - 20b^3 + 20b^2 + 12b^3 - 80b^2 + 80b + 12b^2 - 80b + 80$$

$$= 3b^4 - 8b^3 - 48b^2 + 80$$

$\therefore$ b satisfies $(b+2)^2(3b^2+20b+20) = 0$ A1

(4)

c) 

B1

~~[illegible]~~ Area R_4 = Area R_3 B1

$$\therefore\ \int R_2 + R_4 + R_3 = \int R_2$$

since $R_4 < 0$

(2)

9. Given that $a > b > 0$ and that a and b satisfy the equation

$$\log a - \log b = \log(a - b)$$

(a) show that

$$a = \frac{b^2}{b-1}$$

(3)

(b) Write down the full restriction on the value of b, explaining the reason for this restriction.

(2)

a) $\log a - \log b = \log(a-b)$

$\therefore \log \dfrac{a}{b} = \log(a-b)$ B1

$\therefore \dfrac{a}{b} = a-b$

$\therefore a = ab - b^2$ M1

$\therefore a - ab = -b^2$

$\therefore ab - a = b^2$

$\therefore a(b-1) = b^2$

$\therefore a = \dfrac{b^2}{b-1}$ A1

b) $a > 0 \Rightarrow b > 1$ B1 since $b^2 > 0 \; \forall b$ B1

10. (i) Prove that for all $n \in \mathbb{N}$, $n^2 + 2$ is not divisible by 4

(4)

(ii) "Given $x \in \mathbb{R}$, the value of $|3x - 28|$ is greater than or equal to the value of $(x - 9)$."
State, giving a reason, if the above statement is always true, sometimes true or never true.

(2)

i) Assume n^2+2 is divisible by 4

$\therefore n^2+2 = 4m$ where $m \in \mathbb{N}$

$\therefore n^2 = 4m-2$

$= 2(2m-1)$

$\therefore n^2$ is even

But if n^2 is even, n is even since odd × odd = odd and even × even = even

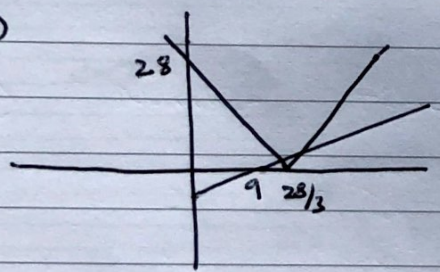
But if n is even, n^2 is a multiple of 4

But if n^2 is a multiple of 4, the next multiple of 4 is n^2+4, not n^2+2

$\times$.

$\therefore n^2+2$ is not divisible by 4 $\forall n \in \mathbb{N}$

ii)



~~if $x = \frac{28}{3}$ $|3x-28| = 0$ and $x-9 = \frac{1}{3}$~~

~~if $x = 0$ $|3x-28| = 28$ $x-9 = -9$~~

~~$\therefore$ sometimes true~~

if $x = 9.4$ $|3x-28| = |28.2-28| = 0.2$

and $x-9 = 0.4$

$\therefore |3x-28| \leq x-9$

if $x = 12$ $|3x-28| = |36-28| = 8$

and $x-9 = 3$

$\therefore |3x-28| \geq x-9$ $\therefore$ sometimes true

11. A competitor is running a 20 kilometre race.

She runs each of the first 4 kilometres at a steady pace of 6 minutes per kilometre.
After the first 4 kilometres, she begins to slow down.

In order to estimate her finishing time, the time that she will take to complete each subsequent kilometre is modelled to be 5% greater than the time that she took to complete the previous kilometre.

Using the model,

(a) show that her time to run the first 6 kilometres is estimated to be 36 minutes 55 seconds, **(2)**

(b) show that her estimated time, in minutes, to run the rth kilometre, for $5 \leqslant r \leqslant 20$, is

$$6 \times 1.05^{r-4}$$

(1)

(c) estimate the total time, in minutes and seconds, that she will take to complete the race. **(4)**

a) Time $= 6 + 6 + 6 + 6 + 6 \times 1.05 + 6 \times 1.05^2$ M1

$= 24 + 6.3 + 6.615 = 36.915$

$= 36$ mins $54\frac{54}{60}$ second

$= 36$ mins 55 seconds A1

b)

Number	Time
1	6
2	6
3	6
4	6
5	6×1.05
6	6×1.05^2
7	6×1.05^3
r	$6 \times 1.05^{r-4}$ B1

c) total time $= 24 + 6 \times 1.05 + 6 \times 1.05^2 + \dots + 6 \times 1.05^{16}$ M1

$= 24 + 6 \times 1.05 \dfrac{(1.05^{16} - 1)}{1.05 - 1}$ M1 A1

$= 173.0421981$

$= 173$ mins $2\frac{31.91}{60}$ seconds

$= 173$ mins 3 seconds

$\frac{1}{}$ second = tierce

Question 12 continued

a) $f'(x) = -2.5e^{-0.25x}\sin x + 10e^{-0.25x}\cos x$ M1 A1

$= 2.5e^{-0.25x}(-\sin x + 4\cos x)$

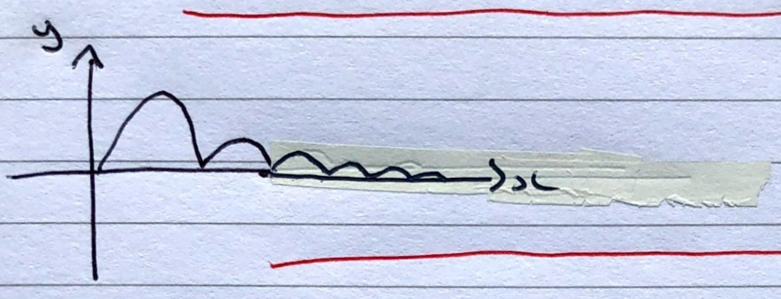
At a turning point, $f'(x)=0$

$\therefore -\sin x + 4\cos x = 0$ M1 (since $2.5e^{-0.25x} \neq 0$)

$\therefore \sin x = 4\cos x$

$\therefore \tan x = 4$ A1 (4)

b)



M1 = 2 loops

A1 (2)

c) Between first and second bounce the equation of the curve is

$g(x) = -10e^{-0.25x}\sin x$

if $g'(x)=0$, then $-2.5e^{-0.25x}(-\sin x + 4\cos x) = 0$

$\therefore \tan x = 4$

$\therefore x = 1.3258,\ 4.4674,\ 7.609$ M1

between first and second bounce, $x = 4.4674$

$\therefore$ Max height $= -10e^{-0.25\times4.4674}\sin 4.4674$ M1 (3)

$= 3.1754$

$= 3.18\text{m}$ (3sf) A1

d) The times between each bounce will not stay the same when the heights are getting smaller. B1 (1)

13. The curve C with equation

$$y = \frac{p - 3x}{(2x - q)(x + 3)} \qquad x \in \mathbb{R}, x \neq -3, x \neq 2$$

where p and q are constants, passes through the point $\left(3, \frac{1}{2}\right)$ and has two vertical asymptotes with equations $x = 2$ and $x = -3$

(a) (i) Explain why you can deduce that $q = 4$

(ii) Show that $p = 15$

(3)

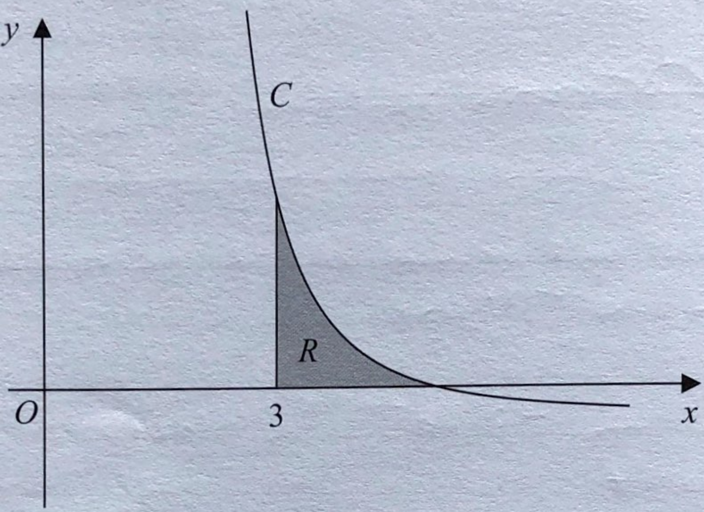


Figure 4

Figure 4 shows a sketch of part of the curve C. The region R, shown shaded in Figure 4, is bounded by the curve C, the x-axis and the line with equation $x = 3$

(b) Show that the exact value of the area of R is $a\ln 2 + b\ln 3$, where a and b are rational constants to be found.

(8)

a) i) since $x=2$ is an asymptote, the denominator equals zero when $x=2$

$\therefore 2x - q = 0 \Rightarrow q = 4$ B1

ii) $x=3, y=\frac{1}{2} \Rightarrow \frac{1}{2} = \frac{p-9}{2\times 6}$ M1

$\therefore 12 \times \frac{1}{2} = p - 9$

$\therefore p - 9 = 6$

$\therefore p = 15$ A1 3

Question 13 continued

b) C cuts the x-axis when $15-3x=0 \Rightarrow x=5$

$$R=\int_3^5 y\,dx$$

$$=\int_3^5 \frac{15-3x}{(2x-4)(x+3)}\,dx$$

$$\frac{15-3x}{2(x-2)(x+3)} \equiv \frac{A}{x-2}+\frac{B}{x+3}$$ M1

$$\therefore \frac{15-3x}{2} \equiv A(x+3)+B(x-2)$$

$x=2 \Rightarrow \frac{9}{2}=5A \Rightarrow A=\frac{9}{10}$

$x=-3 \Rightarrow 12=-5B \Rightarrow B=-\frac{12}{5}$

$$\therefore R=\int_3^5 \frac{9}{10(x-2)}-\frac{12}{5(x+3)}\,dx$$ A1

$$=\left[\frac{9}{10}\ln|x-2|-\frac{12}{5}\ln|x+3|\right]_3^5$$ M1 A1 M1

limits B1

$$=\frac{9}{10}\ln 3-\frac{12}{5}\ln 8-\frac{9}{10}\ln 1+\frac{12}{5}\ln 6$$

$$=\frac{9}{10}\ln 3-\frac{36}{5}\ln 2+\frac{12}{5}\ln 3+\frac{12}{5}\ln 2$$

8

$$=\frac{33}{10}\ln 3-\frac{24}{5}\ln 2$$ correct ln M1

A1

i.e. $a=\frac{33}{10}$, $b=-\frac{24}{5}$

14. The curve C, in the standard Cartesian plane, is defined by the equation

$$x = 4\sin 2y \qquad \frac{-\pi}{4} < y < \frac{\pi}{4}$$

The curve C passes through the origin O

(a) Find the value of $\frac{dy}{dx}$ at the origin.

(2)

(b) (i) Use the small angle approximation for $\sin 2y$ to find an equation linking x and y for points close to the origin.

(ii) Explain the relationship between the answers to (a) and (b)(i).

(2)

(c) Show that, for all points (x, y) lying on C,

$$\frac{dy}{dx} = \frac{1}{a\sqrt{b - x^2}}$$

where a and b are constants to be found.

(3)

a) $x = 4\sin 2y$

$\frac{dx}{dy} = 8\cos 2y$

$\therefore \frac{dy}{dx} = \frac{1}{8\cos 2y}$ M1

At the origin, $y = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{8\cos 0} = \frac{1}{8}$ A1

b) i) For small $2y$, $\sin 2y \simeq 2y$

$\therefore x \simeq 8y$ B1

$\therefore y \simeq \frac{1}{8}x$

The value found in (a) is the gradient of the line found in (b) B1

Question 14 continued

c) $\frac{dy}{dx} = \frac{1}{8\cos 2y}$

but $\sin 2y = \frac{x}{4}$

$\therefore \sin^2 2y = \frac{x^2}{\text{[illegible]}}$

$\therefore \cos^2 2y = 1 - \frac{x^2}{16}$

$\therefore \frac{dy}{dx} = \frac{1}{8\sqrt{1 - \frac{x^2}{16}}}$

$= \frac{1}{\frac{8}{4}\sqrt{16 - x^2}}$ M1 A1

$= \frac{1}{2\sqrt{16 - x^2}}$ A1