

Pure Test 1

① a) At $(k, 8)$, $y = 8$ $\therefore \sec^2 \theta = 8$

$\therefore \sec \theta = 2$

$\therefore \cos \theta = 1/2$

$\therefore \theta = \pi/3$ (MI) (since $0 \leq \theta < \pi/2$)

But $x = k = 3\theta \sin \theta$

$\therefore k = \frac{3\pi \sin^2 \pi/3}{3} = \frac{\pi\sqrt{3}}{2}$ AI (2)

b) $A = \int_0^k y \, dx = \int_0^{\pi/3} \sec^2 \theta \, dx$ (MI)

$x = 3\theta \sin \theta$

$\therefore \frac{dx}{d\theta} = 3\sin \theta + 3\theta \cos \theta$ (MI)

$= \int_0^{\pi/3} (\sec^2 \theta) (3\sin \theta + 3\theta \cos \theta) \, d\theta$

$= \int_0^{\pi/3} \frac{3\sin \theta}{\cos^2 \theta} + \frac{3\theta \cos \theta}{\cos^2 \theta} \, d\theta$

$= \int_0^{\pi/3} \frac{3\sin \theta \sec^2 \theta + 3\theta}{\cos^2 \theta} \, d\theta$

$= 3 \int_0^{\pi/3} \theta \sec^2 \theta + \tan \theta \sec^2 \theta \, d\theta$

$\therefore \alpha = 3, \quad \lambda = 0, \quad \beta = \pi/3$ AI (3)

c) $\int \theta \sec^2 \theta \, d\theta$

$u = \theta \quad \frac{dv}{d\theta} = \sec^2 \theta$

$= \theta \tan \theta - \int \tan \theta \, d\theta$

$\frac{du}{d\theta} = 1 \quad v = \tan \theta$

$= \theta \tan \theta - \ln |\sec \theta| + c_1$ (MIAI)

$\int \tan \theta \sec^2 \theta \, d\theta$

$\frac{d}{d\theta} (\sec^2 \theta) = 2 \sec \theta \sec \theta \tan \theta = 2 \sec^2 \theta \tan \theta$

$= \frac{1}{2} \sec^2 \theta + c_2$ (MIAI)

$\therefore A = 3 \left[\theta \tan \theta - \ln |\sec \theta| + \frac{1}{2} \sec^2 \theta \right]_0^{\pi/3}$

$= 3 \left[\frac{\pi\sqrt{3}}{3} - \ln 2 + 2 - 0 - 0 - \frac{1}{2} \right] = \frac{9}{2} + \pi\sqrt{3} - 3 \ln 2$ AI (5)

② Assume $\sqrt{2}$ is rational B1
 $\therefore \sqrt{2} = \frac{a}{b}$ B1 where a and b are integers with no common factors B1
 $\therefore 2 = \frac{a^2}{b^2}$
 $\therefore a^2 = 2b^2$ B1
 $\therefore a^2$ is an even number which means a is also an even number B1
 So we can write $a = 2n$ where n is an integer B1
 $\therefore (2n)^2 = 2b^2$
 $\therefore 4n^2 = 2b^2$
 $\therefore 2n^2 = b^2$ B1
 $\therefore b^2$ is an even number which means b is also an even number B1
 $\therefore a$ and b have a common factor of 2 B1
 ✗

$\therefore \sqrt{2}$ is irrational B1

③ a) $y = x^x$
 $\therefore \log y = \log x^x$
 $\therefore \log y = x \log x$

Differentiate both sides with respect to x

$$\frac{1}{y} \frac{dy}{dx} = \log x + \frac{x}{x}$$

$$\therefore \frac{dy}{dx} = y (\log x + 1)$$

$$= x^x (\log x + 1)$$

b) $f(1.4) = -0.398$

$f(1.6) = 0.121$

Because there is a change of sign and $f(x)$ is continuous,

$f(x) = x^x - 2$ has a root in the interval $[1.4, 1.6]$ A1

c) $f(x) = x^2 - 2$

$f'(x) = 2x(x+1)$

$x_0 = 1.5$

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)}$ M1A1

$= 1.5 - \frac{-0.16288}{2.5820}$

$= 1.5631$ (4 d.p.) A1

(3)

d) $f(1.55965) = 1.14 \times 10^{-4}$

$f(1.55985) = -1.75 \times 10^{-4}$

Because there is a change of sign and $f(x)$ is continuous,

$x = 1.5596$ (4 d.p.) A1

(2)

(F) a) After 1 bounce, height = $20 \times \frac{1}{2}$

" 2 bounces, " = $20 \times \frac{1}{2}^2$

" " " = $20 \times \frac{1}{2}^n$

$= 5 \times 2^2 \times \frac{1}{2}^n$

$= 5 \times 2^2 \times 2^{-n}$

$= 5 \times 2^{2-n}$

(3)

↑ ↑ ↑ ↑ ... ↑ ↑

④ b) Distance covered = $20 + 20x_{\frac{1}{2}} + 20x_{\frac{1}{2}}^2 + 20x_{\frac{1}{2}}^3 + 20x_{\frac{1}{2}}^4 + \dots$

$$+ \dots + 20x_{\frac{1}{2}}^n + 20x_{\frac{1}{2}}^{n+1}$$

$$= 20 + 2(20x_{\frac{1}{2}} + 20x_{\frac{1}{2}}^2 + \dots + 20x_{\frac{1}{2}}^n)$$

$$= 20 + 2 \left(10 \frac{(1 - \frac{1}{2}^{n+1})}{1 - \frac{1}{2}} \right)$$

$$a = 10$$

$$r = \frac{1}{2}$$

$$n = n + 1$$

$$S_n = a \frac{(1 - r^n)}{1 - r}$$

$$= 20 + 2(20(1 - \frac{1}{2})^{n+1})$$

$$= 20 + 2(20 - 20 \times \frac{1}{2}^{n+1})$$

$$= 60 - 40 \times \frac{1}{2}^{n+1}$$

$$= 60 - 5 \times 8 \times \frac{1}{2}^{n+1}$$

$$= 60 - 5 \times 2^3 \times \frac{1}{2}^{n+1}$$

$$= 60 - 5 \times 2^{4-n}$$

③

④ c) Distance = $20 + 2(20x_{\frac{1}{2}} + 20x_{\frac{1}{2}}^2 + \dots)$

$$a = 10$$

$$r = \frac{1}{2}$$

$$= 20 + 2 \times \frac{10}{1 - \frac{1}{2}}$$

$$S_{\infty} = \frac{a}{1 - r}$$

$$= 20 + 40 = 60$$

④

⑤ a) $u_1 = 4$ $u_2 = k + (-1)^1 u_1 = k - 4$

AI

$$u_3 = k + (-1)^2 u_2 = k + k - 4 = 2k - 4$$

$$u_4 = k + (-1)^3 u_3 = k - (2k - 4) = 4 - k$$

$$u_5 = k + (-1)^4 u_4 = k + 4 - k = 4$$

MAI

b) periodic function. $20/4 = 6 - 2 \therefore u_{26} = u_2 = k - 4$

MAI

c) $4 + k - 4 + 2k - 4 + 4 - k = 6$

MAI

$\therefore 2k = 6$

$\therefore k = 3$

AI

d) $6 \times 6 + 4 + k - 4 = 39$

MAI