Question 1 (**)

The gradient of a curve satisfies

$$\frac{dy}{dx}=\frac{1}{3y^2(x-1)}, \quad x>1.$$

Given the curve passes though the point $P(2,-1)$ and the point $Q(q,1)$, determine the exact value of q.

Question 9 (***)

The area, A km^2, of an oil spillage is growing in time t hours according the differential equation

$$\frac{dA}{dt}=\frac{4e^t}{\sqrt{A}}, \quad A>0.$$

The initial area of the oil spillage was 4 km^2.

a) Solve the differential equation to show that

$$A^3=4\left(3e^t+1\right)^2.$$

b) Find, to three significant figures, the value of t when the area of the spillage reaches 1000 km^2.

Question 15 (***+)

A machine is used to produce waves in the swimming pool of a water theme park.

Let x cm be the height of the wave produced above a certain level in the pool, and suppose it can be modelled by the differential equation

$$\frac{dx}{dt}=2x\sin 2t, \quad t>0,$$

where t is the time in seconds.

When $t=0$, $x=6$.

a) Solve the differential equation to show

$$x=6e^{1-\cos 2t}.$$

b) Find the maximum height of the wave.

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$q=1+e^2$

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$t \approx 8.57$

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b) Find the maximum height of the wave.

$x_{max} \approx 44.3$ cm

Question 5 (***)

A laboratory dish with 100 bacterial cells is placed under observation and 65 minutes later this number has increased to 900 cells.

Let y be the number of bacterial cells present in the dish after t minutes, and assume that y can be treated as a continuous variable.

The rate at which the bacterial cells reproduce is inversely proportional to the square root of the number of the bacterial cells present.

a) Form a differential equation in terms of y, t and a proportionality constant k.

b) Solve the differential equation to show

$$y^{\frac{3}{2}} = At + B,$$

where A and B are constants to be found.

c) Show that the solution to this problem can be written as

$$y^3 = 40000(2t+5)^2.$$

d) Calculate, to the nearest **hour**, the time when the number of bacterial cells reaches 7000.

Question 18 (****)

In a laboratory a dangerous chemical is stored in a cylindrical drum of height 160 cm which is initially full.

One day the drum was found leaking and when this was first discovered, the level of the chemical had dropped to 100 cm, with the level of the chemical dropping at the rate of 0.25 cm per minute.

In order to assess the contamination level in the laboratory, it is required to find the length of time that the leaking has been taking place.

Let h cm be the height of the chemical still left in the drum. It is assumed that the rate at which the height of the chemical is dropping is proportional to the square root of its height.

a) Form a suitable differential equation to model this problem, measuring the time, t minutes, from the instant that the leaking was discovered.

b) Find a solution of this differential equation.

c) Find, in hours and minutes, for how long the leaking has been taking place.

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$\boxed{\frac{dy}{dt} = \frac{k}{\sqrt{y}}}$, $\boxed{A = 400}$, $\boxed{B = 1000}$, $\boxed{t \approx 24}$

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b) Find a solution of this differential equation.

c) Find, in hours and minutes, for how long the leaking has been taking place.

$\boxed{\frac{dh}{dt} = -\frac{1}{40}\sqrt{h}}$, $\boxed{\sqrt{h} = 10 - \frac{1}{80}t \quad \text{or} \quad t = 800 - 80\sqrt{h}}$, $\boxed{\text{3 hours 32 minutes}}$

Question 39 (*****)

A shop stays open for 8 hours every Sunday and its sales, £ x, t hours after the shop opens are modelled as follows.

The rate at which the sales are made, is **directly proportional** to the time left until the shop closes and **inversely proportional** to the sales already made until that time.

Two hours after the shop opens it has made sales worth £336 and sales are made at the rate of £72 per hour.

a) Show clearly that

$$x\frac{dx}{dt} = 4032(8-t).$$

b) Solve the differential equation to show

$$x^2 = 4032t(16-t).$$

c) Find, to the nearest £, the Sunday sales of the shop according to this model.

The shop opens on Sundays at 09.00. The owner knows that the shop is not profitable once the rate at which it makes sales drops under £24 per hour.

d) By squaring the differential equation of part (**a**), find to the nearest minute, the time the shop should close on Sundays.

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