1 Prove that $n^2 - n$ is an even number for all values of n.

2 Prove that $\frac{x}{1+\sqrt{2}} \equiv x\sqrt{2} - x$.

3 Prove that $(x + \sqrt{y})(x - \sqrt{y}) \equiv x^2 - y$.

4 Prove that $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$.

5 Prove that $x^2 + bx \equiv \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

1 Prove that when n is an integer and $1 \leqslant n \leqslant 6$, then $m = n + 2$ is not divisible by 10.

2 Prove that every odd integer between 2 and 26 is either prime or the product of two primes.

3 Prove that the sum of two consecutive square numbers from 1^2 to 8^2 is an odd number.

4 Prove that all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9.

5 Find a counter-example to disprove each of the following statements:

a If n is a positive integer then $n^4 - n$ is divisible by 4.

b Integers always have an even number of factors.

c $2n^2 - 6n + 1$ is positive for all values of n.

d $2n^2 - 2n - 4$ is a multiple of 3 for all integer values of n.

6 A student is trying to prove that $x^3 + y^3 < (x + y)^3$.

The student writes:

> $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
> which is more than $x^3 + y^3$ since
> $3x^2y + 3xy^2 > 0$

a Identify the error made in the proof.

b Provide a counter-example to show that the statement is not true.

7 Prove that for all real values of x

$$(x + 6)^2 \geqslant 2x + 11$$

8 Given that a is a positive real number, prove that:

$$a + \frac{1}{a} \geqslant 2$$

1 $n^2 - n = n(n-1)$
If n is even, $n-1$ is odd and even × odd = even
If n is odd, $n-1$ is even and odd × even = even

2 $\frac{x}{(1+\sqrt{2})} \times \frac{(1-\sqrt{2})}{(1-\sqrt{2})} = \frac{x(1-\sqrt{2})}{(1-2)} = \frac{x - x\sqrt{2}}{-1} = x\sqrt{2} - x$

3 $(x+\sqrt{y})(x-\sqrt{y}) = x^2 - x\sqrt{y} + x\sqrt{y} - y = x^2 - y$

4 $(2x-1)(x+6)(x-5) = (2x-1)(x^2+x-30)$
$= 2x^3 + x^2 - 61x + 30$

5 LHS $= x^2 + bx$, using completing the square,
$\left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$

1 3, 4, 5, 6, 7 and 8 are not divisible by 10
2 3, 5, 7, 11, 13, 17, 19, 23 are prime numbers. 9, 15, 21, 25 are the product of two prime numbers.
3 $1^2 + 2^2 = 5$, $2^2 + 3^2 =$ odd, $3^2 + 4^2 =$ odd, $4^2 + 5^2 =$ odd, $5^2 + 6^2 =$ odd, $6^2 + 7^2 =$ odd, $7^2 + 8^2 = 113$
4 $(3n)^3 = 27n^3 = 9n(3n^2)$ which is a multiple of 9
$(3n+1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9n(3n^2 + 3n + 1) + 1$
which is one more than a multiple of 9
$(3n+2)^3 = 27n^3 + 54n^2 + 36n + 8 = 9n(3n^2 + 6n + 4) + 8$
which is one less than a multiple of 9

5 **a** For example, when $n = 2$, $2^4 - 2 = 14$, 14 is not divisible by 4.
b Any square number
c For example, when $n = \frac{1}{2}$
d For example, when $n = 1$
6 **a** Assuming that x and y are positive
b e.g. $x = 0$, $y = 0$
7 $(x+5)^2 \geqslant 0$ for all real values of x, and
$(x+5)^2 + 2x + 11 = (x+6)^2$, so $(x+6)^2 \geqslant 2x + 11$
8 If $a^2 + 1 \geqslant 2a$ (a is positive, so multiplying both sides by a does not reverse the inequality), then $a^2 - 2a + 1 \geqslant 0$, and $(a-1)^2 \geqslant 0$, which we know is true.
9 **a** $(p+q)^2 = p^2 + 2pq + q^2 = (p-q)^2 + 4pq$
$(p-q)^2 \geqslant 0$ since it is a square, so $(p+q)^2 \geqslant 4pq$
$p > 0, q > 0 \Rightarrow p + q > 0 \Rightarrow p + q \geqslant \sqrt{4pq}$
b e.g. $p = q = -1$: $p + q = -2$, $\sqrt{4pq} = 2$
10 **a** Starts by assuming the inequality is true:
i.e. negative $\geqslant$ positive
b e.g. $x = y = -1$: $x + y = -2$, $\sqrt{x^2 + y^2} = \sqrt{2}$
c $(x+y)^2 = x^2 + 2xy + y^2 > x^2 + y^2$ since $x > 0$, $y > 0 \Rightarrow 2xy > 0$
As $x + y > 0$, can take square roots: $x + y > \sqrt{x^2 + y^2}$