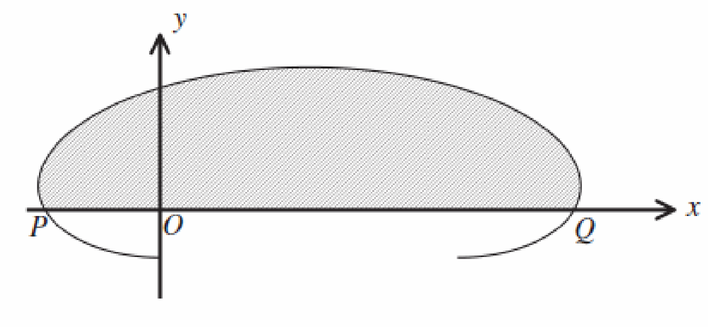
Question 11 (***+)



The figure above shows a curve known as a re-entrant cycloid, with parametric equations

$$x = \theta - 4\sin\theta, \quad y = 1 - 2\cos\theta \,, \quad 0 \le \theta \le 2\pi \,.$$

The curve crosses the x axis at the points P and Q.

a) Find the value of θ at the points P and Q.

b) Show that the area of the finite region bounded by the curve and the x axis, shown shaded in the figure above, is given by the integral

$$\int_{\theta_1}^{\theta_2} 1 - 6\cos\theta + 8\cos^2\theta \; d\theta \,,$$

where θ_1 and θ_2 must be stated.

c) Find an exact value for the above integral.

$\boxed{\theta = \frac{\pi}{3}, \frac{5\pi}{3}}$, $\boxed{\theta_1 = \frac{\pi}{3}, \theta_2 = \frac{5\pi}{3}}$, $\boxed{\frac{20\pi}{3} + 4\sqrt{3}}$

Question 6 (**+)

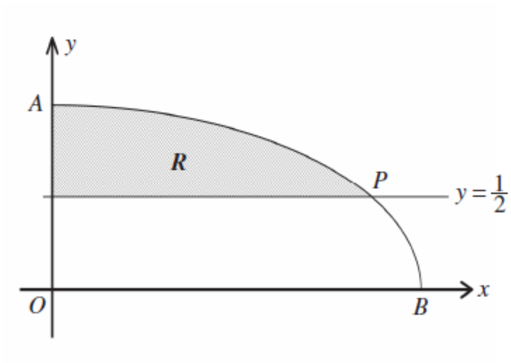
$$f(x) = \frac{2x}{(1+2x)^3}, \quad x \ne -\tfrac{1}{2}.$$

a) Find the first 4 terms in the series expansion of $f(x)$.

b) State the range of values of x for which the expansion of $f(x)$ is valid.

$\boxed{}$, $\boxed{f(x) = 2x - 12x^2 + 48x^3 - 160x^4 + O(x^5)}$, $\boxed{-\tfrac{1}{2} < x < \tfrac{1}{2}}$

Question 16 (****)



The figure above shows the curve C, with parametric equations

$$x = 4\cos\theta, \quad y = \sin\theta \ , \ 0 \le \theta \le \frac{\pi}{2}.$$

The curve meets the coordinate axes at the points A and B. The straight line with equation $y = \frac{1}{2}$ meets C at the point P.

a) Show that the area under the arc of the curve between A and P, and the x axis, is given by the integral

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 4\sin^2\theta \ d\theta .$$

The shaded region R is bounded by C, the straight line with equation $y = \frac{1}{2}$ and the y axis.

b) Find an exact value for the area of R.

$\boxed{\text{area} = \frac{1}{6}\left(4\pi - 3\sqrt{3}\right)}$

4. $\int -3x\cos 4x \ dx = -\frac{3}{4}x\sin 4x - \frac{3}{16}\cos 4x + C$

5. $\int x^2 e^{-2x} \ dx = -\frac{1}{2}x^2 e^{-2x} - \frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C$

6. $\int x^2 \sin 5x \ dx = -\frac{1}{5}x^2\cos 5x + \frac{2}{25}x\sin 5x + \frac{2}{125}\cos 5x + C$

7. $\int x^2 \cos\tfrac{1}{3}x \ dx = 3x^2\sin\tfrac{1}{3}x + 18x\cos\tfrac{1}{3}x - 54\sin\tfrac{1}{3}x + C$

Question 7 (***)

$$x = 4\sin\theta + 7\cos\theta .$$

The value of θ is increasing at the constant rate of 0.5, in suitable units.

Find the rate at which x is changing, when $\theta = \frac{\pi}{2}$.

$\boxed{}$, $\boxed{-\frac{7}{2}}$

Question 17

$$f(x) \equiv 2\sin x + 2\cos x,\ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R > 0$, $0 < \alpha < \frac{\pi}{2}$.

b) State the minimum and the maximum value of …

i. … $y = f\left(x - \frac{\pi}{2}\right)$.

ii. … $y = 2f(x)+1$.

iii. … $y = [f(x)]^2$.

iv. … $y = \frac{10}{f(x)+3\sqrt{2}}$.

$\boxed{f(x) \equiv \sqrt{8}\sin\left(x+\frac{\pi}{4}\right)}$, $\boxed{[-\sqrt{8}, \sqrt{8}]}$, $\boxed{[-2\sqrt{8}+1, 2\sqrt{8}+1]}$, $\boxed{[0,8]}$, $\boxed{[\sqrt{2}, 5\sqrt{2}]}$

Question 10 (***)

Liquid dye is poured onto a large flat cloth and forms a circular stain, the area of which grows at a steady rate of $1.5 \text{ cm}^2\text{s}^{-1}$.

Calculate, correct to three significant figures, …

a) … the radius, in cm, of the stain 4 seconds after it started forming.

b) … the rate, in cm s^{-1}, of increase of the radius of the stain after 4 seconds.

$\boxed{}$, $\boxed{r = \sqrt{\frac{6}{\pi}} \approx 1.38 \text{ cm}}$, $\boxed{\sqrt{\frac{3}{32\pi}} \approx 0.173 \text{ cm s}^{-1}}$

Question 11 (***)

The variables y, x and t are related by the equations

$$y = 15\left(4 - \frac{27}{(x+3)^3}\right) \quad \text{and} \quad \ln(x+3) = \frac{1}{3}t, \; x > -3.$$

Find the value of $\frac{dy}{dt}$, when $x = 9$.

$\boxed{}$, $\boxed{\frac{dy}{dt} = \frac{15}{64}}$

Question 13 (***)

$$f(x) = \sqrt{1-2x}, \quad |x| < \frac{1}{2}.$$

a) Expand $f(x)$ as an infinite series, up and including the term in x^3.

b) By substituting $x = 0.01$ in the expansion, show that $\sqrt{2} \approx 1.414214$.

$\boxed{f(x) = 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + O(x^4)}$

Question 14 (****)

Liquid is pouring into a container at the constant rate of $30\ \text{cm}^3\text{s}^{-1}$.

The container is initially empty and when the height of the liquid in the container is h cm the volume of the liquid, V cm^3, is given by

$$V = 36h^2.$$

a) Find the rate at which the height of the liquid in the container is rising when the height of the liquid reaches 3 cm.

b) Determine the rate at which the height of the liquid in the container is rising 12.5 minutes after the liquid started pouring in.

$\boxed{}$, $\boxed{\frac{5}{36} = 0.139\ \text{cms}^{-1}}$, $\boxed{\frac{1}{60} = 0.0167\ \text{cms}^{-1}}$