Differential Equations 3

1 The rate of increase of a population P of rabbits at time t, in years, is given by $\frac{dP}{dt} = kP$, $k > 0$.
Initially the population was of size 200.

a Solve the differential equations giving P in terms of k and t. **(3 marks)**

b Given that $k = 3$, find the time taken for the population to reach 4000. **(4 marks)**

c State a limitation of this model for large values of t. **(1 mark)**

2 A mug of tea, with a temperature T °C is made and left to cool in a room with a temperature of 25 °C. The rate at which the tea cools is proportional to the difference in temperature between the tea and the room.

a Show that this process can be described by the differential equation $\frac{dT}{dt} = -k(T - 25)$, explaining why k is a positive constant. **(3 marks)**

Initially the tea is at a temperature of 85 °C. 10 minutes later the tea is at 55 °C.

b Find the temperature, to 1 decimal place, of the tea after 15 minutes. **(7 marks)**

3 The rate of change of the surface area of a drop of oil, A mm^2, at time t minutes can be modelled by the equation $\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{10t^2}$

Given that the surface area of the drop is 1 mm^2 at $t = 1$,

a find an expression for A in terms of t **(7 marks)**

b show that the surface area of the drop cannot exceed $\frac{400}{361}$ mm^2. **(2 marks)**

4

Liquid is pouring into a container at a constant rate of $40\text{ cm}^3\text{ s}^{-1}$ and is leaking from the container at a rate of $\frac{1}{4}V\text{ cm}^3\text{ s}^{-1}$, where $V\text{ cm}^3$ is the volume of liquid in the container.

a Show that $-4\frac{dV}{dt} = V - 160$. **(2 marks)**

Given that $V = 5000$ when $t = 0$,

b find the solution to the differential equation in the form $V = a + be^{-\frac{1}{4}t}$, where a and b are constants to be found **(7 marks)**

c write down the limiting value of V as $t \to \infty$. **(1 mark)**

5

a Express $\frac{1}{P(10\,000 - P)}$ using partial fractions. **(3 marks)**

The deer population, P, in a reservation can be modelled by the differential equation

$$\frac{dP}{dt} = \frac{1}{200}P(10\,000 - P)$$

where t is the time in years since the study began. Given that the initial deer population is 2500,

b solve the differential equation giving your answer in the form $P = \frac{a}{b + ce^{-50t}}$ **(6 marks)**

c Find the maximum deer population according to the model. **(2 marks)**

6

Fossils are aged using a process called carbon dating. The amount of carbon remaining in a fossil, R, decreases over time, t, measured in years. The rate of decrease of carbon is proportional to the remaining carbon.

a Given that initially the amount of carbon is R_0, show that $R = R_0e^{-kt}$ **(4 marks)**

It is known that the half-life of carbon is 5730 years. This means that after 5730 years the amount of carbon remaining has reduced by half.

b Find the exact value of k. **(3 marks)**

c A fossil is found with 10% of its expected carbon remaining. Determine the age of the fossil to the nearest year. **(3 marks)**