Version A Nov 2018

A2 Tracking Test 2 PURE (out of 57)

70 minutes

1. *(a)* Use integration by parts to find:-

$$\int xe^{2x-1}\,dx$$

(3)

(*b*) Use integration by parts to find $\int \frac{1}{x^3} \ln x \, dx$.

(4)

2. (*a*) For $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, sketch the graph of $y = g(x)$ where

$$g(x) = \arcsin x, \qquad -1 \leq x \leq 1.$$

(2)

(*b*) Find the exact value of x for which

$$3g(x + 1) + \pi = 0.$$

(3)

3.

A curve has equation $3x^2 - y^2 + xy = 4$. The points P and Q lie on the curve. The gradient of the tangent to the curve is $\frac{8}{3}$ at P and at Q.

(a) Use implicit differentiation to show that $y - 2x = 0$ at P and at Q.

(6)

(b) Find the coordinates of P and Q.

(3)

4

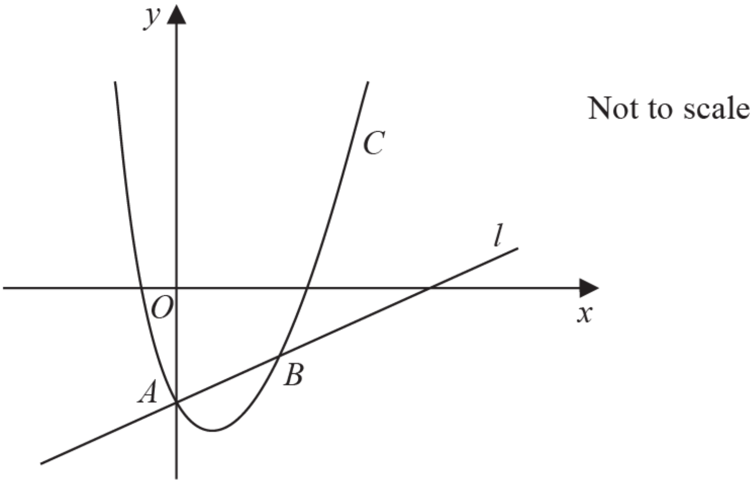


Figure 1

Figure 1 shows a sketch of part of the curve C with equation

$$y = e^{-2x} + x^2 - 3$$

The curve C crosses the y-axis at the point A.

The line l is the normal to C at the point A.

(a) Find the equation of l, writing your answer in the form $y = mx + c$, where m and c are constants. **(5)**

The line l meets C again at the point B, as shown in Figure 1.

(b) Show that the x coordinate of B is a solution of

$$x = \sqrt{1 + \frac{1}{2}x - e^{-2x}}$$

(2)

Using the iterative formula

$$x_{n+1} = \sqrt{1 + \frac{1}{2}x_n - e^{-2x_n}}$$

with $x_1 = 1$

(c) find x_2 and x_3 to 3 decimal places. **(2)**

5 (i) Using the identity for tan ($A \pm B$), solve, for $-90° < x < 90°$,

$$\frac{\tan 2x + \tan 32°}{1 - \tan 2x \tan 32°} = 5$$

Give your answers, in degrees, to 2 decimal places. **(4)**

(ii) (*a*) Using the identity for tan ($A \pm B$), show that

$$\tan(3\theta - 45°) \equiv \frac{\tan 3\theta - 1}{1 + \tan 3\theta}, \qquad \theta \neq (60n + 45)°,\ n \in \mathbb{Z}$$

(2)

(*b*) Hence solve, for $0 < \theta < 180°$,

$$(1 + \tan 3\theta)\tan(\theta + 28°) = \tan 3\theta - 1$$

(4)

6.

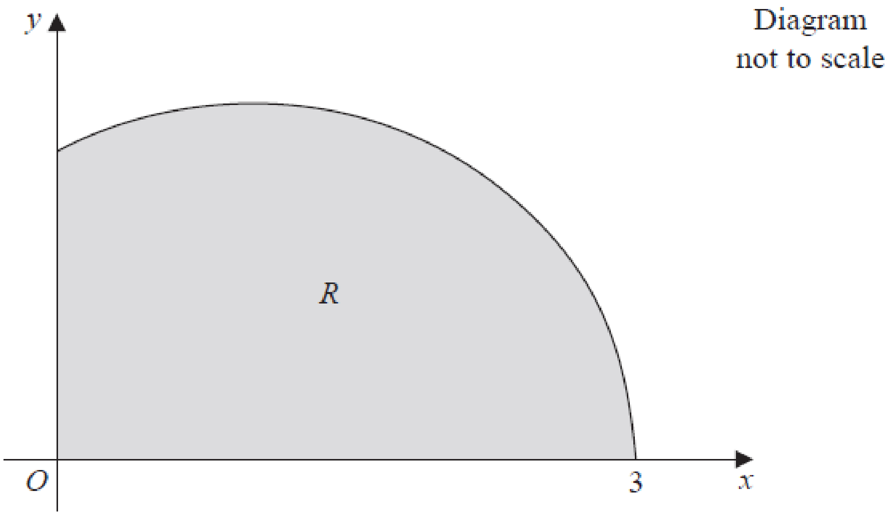


Figure 2

Figure 2 shows a sketch of the curve with equation $y = \sqrt{(3-x)(x+1)}$, $0 \leq x \leq 3$.

The finite region R, shown shaded in Figure 2, is bounded by the curve, the x-axis, and the y-axis.

(a) Use the substitution $x = 1 + 2\sin\theta$ to show that

$$\int_0^3 \sqrt{(3-x)(x+1)}\ \mathrm{d}x = k\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\ \mathrm{d}\theta,$$

where k is a constant to be determined.

(5)

(b) Hence find, by integration, the exact area of R.

(3)

7

a) Express $2sin\theta - 4cos\theta$ in the form $Rsin(\theta - \alpha)$, where R and α are constants, R > 0 and $0 < \frac{\alpha}{2} < \frac{\pi}{2}$. Give the exact value of R and the value of α, in radians to 3 decimal places. (3)

$$H(\theta) = 4 + 5(2sin3\theta - 4cos3\theta)^2$$

Find:

b) i) the maximum value of $H(\theta)$

ii) the smallest value of θ, for $0 \leq \theta \leq \pi$, at which this maximum occurs. (3)

Find:

c) i) the minimum value of $H(\theta)$

ii) the largest value of θ, for $0 \leq \theta \leq \pi$, at which this minimum occurs (3)

<u>Mark Scheme (pure)</u>

1.

(a)	$I = \int xe^{2x-1}\mathrm{d}x$	
	$u = x \qquad \frac{dv}{dx} = e^{2x-1}$	
	$\frac{du}{dx} = 1 \qquad v = \frac{1}{2}e^{2x-1}$	
	$I = \frac{1}{2}xe^{2x-1} - \int \frac{1}{2}e^{2x-1}dx$	M1, A1
	$I = \frac{1}{2}xe^{2x-1} - \frac{1}{4}e^{2x-1} + C$	A1
		[3]

b/ $\int \frac{1}{x^3} \ln x \, dx$

$u = \ln x \qquad \frac{dv}{dx} = x^{-3}$

$\frac{dv}{dx} = \frac{1}{x} \qquad v = -\frac{1}{2}x^{-2}$

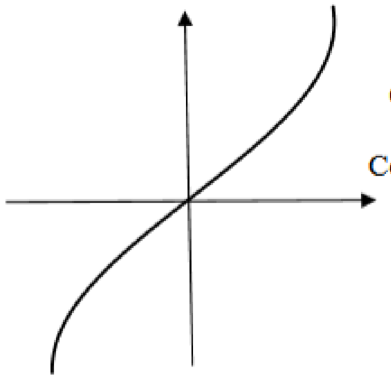
$-\frac{1}{2}x^{-2}\ln x - \int -\frac{1}{2}x^{-3}\,dx$

$-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C$

M1 for seperating into a u and derivaive of v. A1 for correct table.

M1 for using parts formula. A1 for correct answer.

2.

(a)		Correct position **or** curvature	M1
		Correct position **and** curvature	A1
			(2)
(b)	$3\arcsin(x+1)+\pi=0 \Rightarrow \arcsin(x+1)=-\frac{\pi}{3}$		M1
	$\Rightarrow (x+1)=\sin\left(-\frac{\pi}{3}\right)$		
	$\Rightarrow x=-1-\frac{\sqrt{3}}{2}$		dM1A1
			(3)
			(5 marks)

3

(a) $3x^2 - y^2 + xy = 4$ (eqn *)

$$\underline{6x - 2y\frac{dy}{dx}} + \left(\underline{y + x\frac{dy}{dx}}\right) = \underline{0}$$

$$\left\{\frac{dy}{dx} = \frac{-6x-y}{x-2y}\right\} \text{ or } \left\{\frac{dy}{dx} = \frac{6x+y}{2y-x}\right\}$$ *not necessarily required.*

$$\frac{dy}{dx} = \frac{8}{3} \Rightarrow \frac{-6x-y}{x-2y} = \frac{8}{3}$$

giving $-18x - 3y = 8x - 16y$

giving $13y = 26x$

Hence, $y = 2x \Rightarrow \underline{y - 2x = 0}$

Differentiates implicitly to include either

$\pm ky\frac{dy}{dx}$ or $x\frac{dy}{dx}$. (Ignore $\left(\frac{dy}{dx}=\right)$) M1

Correct application $\underline{\underline{(\ )}}$ of product rule B1

$(3x^2 - y^2) \rightarrow \underline{\left(6x - 2y\frac{dy}{dx}\right)}$ and $(4 \rightarrow \underline{0})$ A1

Substituting $\frac{dy}{dx} = \frac{8}{3}$ into their equation. M1*

Attempt to combine either terms in x or terms in y together

to give either ax or by. dM1*

simplifying to give $\underline{y - 2x = 0}$ **AG** A1 **cso** 6

(b) At P & Q, $y = 2x$. Substituting into eqn *
gives $3x^2 - (2x)^2 + x(2x) = 4$
Simplifying gives, $x^2 = 4 \Rightarrow \underline{x = \pm 2}$
$y = 2x \Rightarrow y = \pm 4$
Hence coordinates are $\underline{(2, 4)}$ and $\underline{(-2, -4)}$

Attempt replacing y by $2x$ **in at least one** of the y terms in eqn * M1

Either $x = 2$ or $x = -2$ A1

Both (2, 4) and (−2, −4) A1 3

[9]

4.

4.(a)	$\frac{dy}{dx} = -2e^{-2x} + 2x$	M1A1	
	At $x = 0$ $\frac{dy}{dx} = -2 \Rightarrow \frac{dx}{dy} = \frac{1}{2}$	M1	
	Equation of normal is $y - (-2) = \frac{1}{2}(x - 0) \Rightarrow y = \frac{1}{2}x - 2$	M1 A1	
			(5)
(b)	$y = e^{-2x} + x^2 - 3$ meets $y = \frac{1}{2}x - 2$ when $e^{-2x} + x^2 - 3 = "\frac{1}{2}x - 2"$		
	$x^2 = 1 + \frac{1}{2}x - e^{-2x}$	M1	

	$x=\sqrt{1+\frac{1}{2}x-e^{-2x}}$ *	A1*
		(2)
(c)	$x_2=\sqrt{1+0.5-e^{-2}}$	M1
	$x_2=1.168,\ x_3=1.220$	A1
		(2)
		(9 marks)

5.

(i)	$\frac{\tan 2x+\tan 32°}{1-\tan 2x\tan 32°}=5\Rightarrow\tan(2x+32°)=5$	B1
	$\Rightarrow x=\frac{\arctan 5-32°}{2}$	M1
	$\Rightarrow x=\text{awrt } 23.35°, -66.65°$	A1A1
		(4)
(ii)(a)	$\tan(3\theta-45°)=\frac{\tan 3\theta-\tan 45°}{1+\tan 45°\tan 3\theta}=\frac{\tan 3\theta-1}{1+\tan 3\theta}$	M1A1*
		(2)
(b)	$(1+\tan 3\theta)\tan(\theta+28°)=\tan 3\theta-1$	
	$\Rightarrow\tan(\theta+28°)=\tan(3\theta-45°)$	B1
	$\theta+28°=3\theta-45°\Rightarrow\theta=36.5°$	M1A1
	$\theta+28°+180°=3\theta-45°\Rightarrow\theta=126.5°$	A1
		(4)
		(10 marks)

6.

(a)	$A = \int_0^3 \sqrt{(3-x)(x+1)}\,\mathrm{d}x$, $x = 1 + 2\sin\theta$		
	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$	$\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\cos\theta$ or $2\cos\theta$ used correctly in their working. Can be implied.	B1
	$\left\{\int \sqrt{(3-x)(x+1)}\,\mathrm{d}x \text{ or } \int \sqrt{(3+2x-x^2)}\,\mathrm{d}x\right\}$		
	$= \int \sqrt{(3-(1+2\sin\theta))((1+2\sin\theta)+1)}\;2\cos\theta\,\{\mathrm{d}\theta\}$	Substitutes for both x and $\mathrm{d}x$, where $\mathrm{d}x \neq \lambda\mathrm{d}\theta$. Ignore $\mathrm{d}\theta$	M1
	$= \int \sqrt{(2-2\sin\theta)(2+2\sin\theta)}\;2\cos\theta\,\{\mathrm{d}\theta\}$ $= \int \sqrt{(4-4\sin^2\theta)}\;2\cos\theta\,\{\mathrm{d}\theta\}$		
	$= \int \sqrt{(4-4(1-\cos^2\theta)}\;2\cos\theta\,\{\mathrm{d}\theta\}$ or $\int \sqrt{4\cos^2\theta}\;2\cos\theta\,\{\mathrm{d}\theta\}$	Applies $\cos^2\theta = 1 - \sin^2\theta$ **see notes**	M1
	$= 4\int \cos^2\theta\,\mathrm{d}\theta,\ \{k = 4\}$	$4\int \cos^2\theta\,\mathrm{d}\theta$ or $\int 4\cos^2\theta\,\mathrm{d}\theta$ **Note:** $\mathrm{d}\theta$ is required here.	A1
	$0 = 1 + 2\sin\theta$ or $-1 = 2\sin\theta$ or $\sin\theta = -\frac{1}{2} \Rightarrow \underline{\theta = -\frac{\pi}{6}}$ **and** $3 = 1 + 2\sin\theta$ or $2 = 2\sin\theta$ or $\sin\theta = 1 \Rightarrow \underline{\theta = \frac{\pi}{2}}$	See notes	B1
			[5]
(b)	$\left\{k\int \cos^2\theta\,\{\mathrm{d}\theta\}\right\} = \{k\}\int\left(\frac{1+\cos 2\theta}{2}\right)\{\mathrm{d}\theta\}$	Applies $\cos 2\theta = 2\cos^2\theta - 1$ to their integral	M1
	$= \{k\}\left(\frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)$	Integrates to give $\pm\alpha\theta \pm \beta\sin 2\theta$, $\alpha \neq 0$, $\beta \neq 0$ or $k(\pm\alpha\theta \pm \beta\sin 2\theta)$	M1 **(A1 on ePEN)**
	$\left\{\text{So } 4\int_{-\frac{\pi}{6}}^{\frac{\pi}{2}} \cos^2\theta\,\mathrm{d}\theta = \left[2\theta + \sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{2}}\right\}$ $= \left(2\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{2}\right)\right) - \left(2\left(-\frac{\pi}{6}\right) + \sin\left(-\frac{2\pi}{6}\right)\right)$		
	$\left\{= (\pi) - \left(-\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right\} = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$	$\frac{4\pi}{3} + \frac{\sqrt{3}}{2}$ **or** $\frac{1}{6}\left(8\pi + 3\sqrt{3}\right)$	A1 **cao cso**
			[3]
			8

7.

9.(a)	$R = \sqrt{20}$	B1
	$\tan\alpha = \frac{4}{2} \Rightarrow \alpha = \text{awrt } 1.107$	M1A1
		(3)
(b)(i)	$'4 + 5R^2' = 104$	B1ft
(ii)	$3\theta - '1.107' = \frac{\pi}{2} \Rightarrow \theta = \text{awrt } 0.89$	M1A1
		(3)
(c)(i)	4	B1
(ii)	$3\theta - '1.107' = 2\pi \Rightarrow \theta = \text{awrt } 2.46$	M1A1
		(3)
		(9 marks)