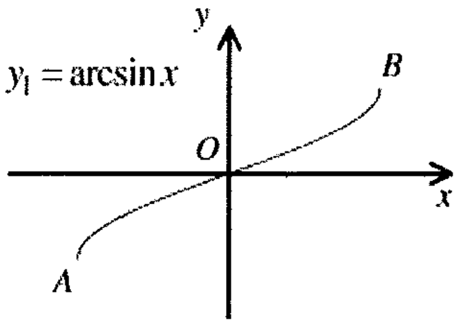
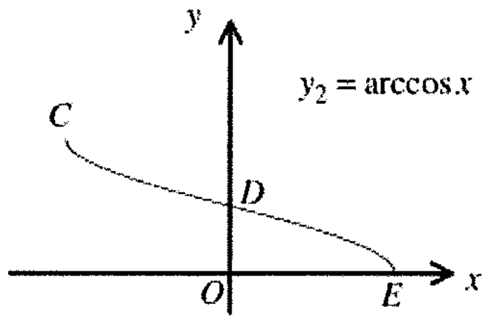
Revision for Tracking test 2

1 $\displaystyle\int \frac{1}{2} x e^{4x}\, dx$

2





The diagrams above shows the graphs of $y_1 = \arcsin x$ and $y_2 = \arccos x$.

The graph of y_1 has endpoints at A and B.

The graph of y_2 has endpoints at C and E, and D is the point where the graph of y_2 crosses the y axis.

State the coordinates of A, B, C, D and E.

3 Solve the following trigonometric equation

$$3\operatorname{arccot}\left(x-\sqrt{3}\right)-\pi=0.$$

4 For the following implicit relationship, find an expression for $\frac{dy}{dx}$, in terms of x and y.

$$e^{2x}+e^{2y}=xy$$

5. Differentiate each of the following expressions with respect to x, simplifying the final answers where possible.

a) $y = \dfrac{1}{\sqrt{1-2x}}$.

b) $y = e^{3x}(\sin x + \cos x)$.

c) $y = \dfrac{\ln x}{x^2}$.

6. Prove $\quad 4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta \equiv \sec^2\theta$

7.
$$f(x) \equiv 2\sqrt{2}\cos x + 2\sqrt{2}\sin x\,,\ x \in \mathbb{R}.$$

a) Express $f(x)$ in the form $R\sin(x+\alpha)$, $R > 0$, $0 < \alpha < \dfrac{\pi}{2}$.

b) Solve the trigonometric equation

$$f(x) = 2 \text{ for } 0 < x < 2\pi.$$

c) Write down the maximum value of $f(x)$.

d) Find the smallest positive value of x for which this maximum value occurs.

(8)

$$x^3 - x^2 = 6x + 6, \quad x \in \mathbb{R}.$$

a) Show that the above equation has a root α in the interval $(3, 4)$.

b) Show that the above equation can be written as

$$x = \sqrt{\frac{6x+6}{x-1}}.$$

An iterative formula of the form given in part (b), starting with x_0 is used to find α.

c) Give two different values for x_0 that would not produce an answer for x_1.

d) Starting with $x_0 = 3.3$ find the value of x_1, x_2, x_3 and x_4, giving each of the answers correct to 3 decimal places.

e) By considering the sign of an appropriate function in a suitable interval, show clearly that $\alpha = 3.33691$, correct to 5 decimal places.

(9) By expanding $\sin(45° - x)$ with a suitable value for x, show clearly that

$$\operatorname{cosec} 15° = \sqrt{2} + \sqrt{6}.$$

(10) $\displaystyle\int_0^2 \frac{1}{(3x^2+4)^{\frac{3}{2}}}\,dx$: use $x = \frac{2}{\sqrt{3}} \tan\theta$

1. $\int \tfrac{1}{2} x e^{4x} \, dx = \frac{1}{8} x e^{4x} - \frac{1}{32} e^{4x} + C$

2 $\boxed{A\left(-1, \frac{\pi}{2}\right)}$, $\boxed{B\left(1, \frac{\pi}{2}\right)}$, $\boxed{C(-1, \pi)}$, $\boxed{D\left(0, \frac{\pi}{2}\right)}$, $\boxed{E(1,0)}$,

3 $\boxed{x = \tfrac{4}{3}\sqrt{3}}$

4 $\boxed{\frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}}$,

5 $\boxed{\frac{dy}{dx} = (1-2x)^{-\frac{3}{2}}}$, $\boxed{\frac{dy}{dx} = 2e^{3x}(\sin x + 2\cos x)}$, $\boxed{\frac{dy}{dx} = \frac{1 - 2\ln x}{x^3}}$

7 $\boxed{f(x) \equiv 4\sin\left(x + \frac{\pi}{4}\right)}$, $\boxed{x = \frac{7\pi}{12}, \frac{23\pi}{12}}$, $\boxed{f(x)_{\max} = 4}$, $\boxed{x = \frac{\pi}{4}}$

8 . $\boxed{x_0 \neq 1,\ 0,\ 0.5 \text{ etc}}$, $\boxed{x_1 = 3.349,\ x_2 = 3.333,\ x_3 = 3.338,\ x_4 = 3.336}$

10 $\int_0^2 \frac{1}{(3x^2+4)^{\frac{3}{2}}} \, dx = \frac{1}{8}$, use $x = \frac{2}{\sqrt{3}} \tan\theta$

(1) $\displaystyle\int \frac{1}{2} x e^{4x}\,dx$ $\qquad u = x \qquad \dfrac{dv}{dx} = e^{4x}$

$$= \frac{1}{2}\int x e^{4x}\,dx \qquad \frac{du}{dx} = 1 \qquad v = \frac{1}{4}e^{4x}$$

$$= \frac{1}{2}\left[\frac{1}{4} x e^{4x} - \int \frac{1}{4} e^{4x}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} x e^{4x} - \frac{1}{16} e^{4x}\right] + c$$

$$= \frac{1}{8} x e^{4x} - \frac{1}{32} e^{4x} + c$$

(2) $A(-1, \pi/2) \quad B(1, \pi/2) \quad C(-1, \pi) \quad D(0, \pi/2) \quad E(1, 0)$

(3) $3 \operatorname{arccot}(x - \sqrt{3}) - \pi = 0$

$\therefore 3 \operatorname{arccot}(x - \sqrt{3}) = \pi$

$\therefore \operatorname{arccot}(x - \sqrt{3}) = \dfrac{\pi}{3}$

$\therefore x - \sqrt{3} = \cot \pi/3$

$\therefore x - \sqrt{3} = \dfrac{1}{\tan \pi/3}$

$\therefore x - \sqrt{3} = \dfrac{1}{\sqrt{3}}$

$\therefore x - \sqrt{3} = \dfrac{\sqrt{3}}{3}$

$\therefore x = \dfrac{\sqrt{3}}{3} + \sqrt{3}$

$\therefore x = \sqrt{3}\left(\frac{1}{3} + 1\right)$

$\therefore x = \dfrac{4\sqrt{3}}{3}$

(4) $e^{2x} + e^{2y} = xy$

Differentiate b.s. w.r.t. x

$\therefore \quad 2e^{2x} + 2e^{2y}\frac{dy}{dx} = y + x\frac{dy}{dx}$

$\therefore \quad \frac{dy}{dx}(2e^{2y} - x) = y - 2e^{2x}$

$\therefore \quad \frac{dy}{dx} = \frac{y - 2e^{2x}}{2e^{2y} - x}$

(5) a) $y = (1-2x)^{-1/2}$

$\therefore \frac{dy}{dx} = -\frac{1}{2}(1-2x)^{-3/2}(-2)$

$= (1-2x)^{-3/2}$

b) $y = e^{3x}(\sin x + \cos x)$

$\therefore \frac{dy}{dx} = 3e^{3x}(\sin x + \cos x) + e^{3x}(\cos x - \sin x)$

$= e^{3x}(3\sin x + 3\cos x + \cos x - \sin x)$

$= e^{3x}(2\sin x + 4\cos x)$

$= 2e^{3x}(\sin x + 2\cos x)$

c) $y = \frac{\ln x}{x^2}$

$\therefore \frac{dy}{dx} = \frac{(\frac{1}{x})(x^2) - (2x)(\ln x)}{x^4} = \frac{x - 2x\ln x}{x^4}$

$= \frac{x(1 - 2\ln x)}{x^4} = \frac{1 - 2\ln x}{x^3}$

(6) $LHS \equiv 4\operatorname{cosec}^2 2\theta - \operatorname{cosec}^2\theta$

$$\equiv \frac{4}{(\sin 2\theta)^2} - \frac{1}{\sin^2\theta}$$

$$\equiv \frac{4}{(2\sin\theta\cos\theta)^2} - \frac{1}{\sin^2\theta}$$

$$\equiv \frac{1}{\sin^2\theta\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$\equiv \frac{1}{\sin^2\theta\cos^2\theta} - \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\equiv \frac{1-\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\equiv \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta}$$

$$\equiv \frac{1}{\cos^2\theta}$$

$\equiv \sec^2\theta \equiv RHS$ As required

Proof complete

QED

□

(7) a) $f(x) = 2\sqrt{2}\cos x + 2\sqrt{2}\sin x$

$\sqrt{(2\sqrt{2})^2 + (2\sqrt{2})^2} = \sqrt{16} = 4$

$\therefore f(x) = 4\left(\frac{2\sqrt{2}}{4}\cos x + \frac{2\sqrt{2}}{4}\sin x\right)$

$= 4\left(\frac{\sqrt{2}}{2}\cos x + \frac{\sqrt{2}}{2}\sin x\right)$

$R\sin(x+\alpha) = R(\sin x\cos\alpha + \cos x\sin\alpha)$

$\therefore \cos\alpha = \frac{\sqrt{2}}{2} \Rightarrow \alpha = \frac{\pi}{4}$

$\therefore f(x) = 4\sin(x + \pi/4)$

b) $f(x) = 2$

$\Rightarrow 4\sin(x + \pi/4) = 2$

$\therefore \sin(x + \pi/4) = \frac{1}{2}$

$\therefore x + \pi/4 = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \ldots$

$\therefore x = \frac{7\pi}{12}, \frac{23\pi}{12}$ since $0 < x < 2\pi$

c) 4

d) $4\sin(x + \pi/4) = 4$

$\Rightarrow \sin(x + \pi/4) = 1$

$\Rightarrow x + \pi/4 = \pi/2, \ldots$

$\Rightarrow x = \frac{\pi}{4}$

(8) a) Let $f(x) = x^3 - x^2 - 6x - 6$

$f(3) = 27 - 9 - 18 - 6$
$= -6$

$f(4) = 64 - 16 - 24 - 6$
$= 18$

Since there is a change of sign and $f(x)$ is continuous,
$x^3 - x^2 = 6x + 6$ has a root α in the interval $(3, 4)$

b) $x^3 - x^2 - 6x - 6 = 0$

$\Rightarrow x^2(x-1) = 6x + 6$

$\Rightarrow x^2 = \frac{6x+6}{x-1}$

$\Rightarrow x = \sqrt{\frac{6x+6}{x-1}}$

c) $x_0 = 1$ would not produce an answer for x_1 since the denominator would be zero

$x_0 = 0$ also would not produce an answer for x_1, since the expression inside the square root sign would be negative

d) $x_0 = 3.3$, $x_1 = 3.349$, $x_2 = 3.333$, $x_3 = 3.338$, $x_4 = 3.336$

e) $f(3.336905) = -0.000145$
$f(3.336915) = 0.0000625$

Since there is a change of sign and $f(x)$ is continuous
$\alpha = 3.33691$ (5 d.p.)

(9) $\sin(45-x) = \sin 45 \cos x - \cos 45 \sin x$

$\operatorname{cosec} 15 = \dfrac{1}{\sin 15} = \dfrac{1}{\sin(45-30)}$

$\therefore$ if $x = 30$, $\sin(45-x) = \sin 45 \cos 30 - \cos 45 \sin 30$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore \operatorname{cosec} 15 = \frac{2\sqrt{2}}{\sqrt{3}-1} = \frac{2\sqrt{2}(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}$$

$$= \frac{2\sqrt{6}+2\sqrt{2}}{3-1}$$

$$= \sqrt{6}+\sqrt{2}$$

(10) $$\int_0^2 \frac{1}{(3x^2+4)^{3/2}}\,dx$$

$x = \dfrac{2}{\sqrt{3}}\tan\theta$

$\dfrac{dx}{d\theta} = \dfrac{2}{\sqrt{3}}\sec^2\theta$

$\therefore dx = \dfrac{2}{\sqrt{3}}\sec^2\theta\, d\theta$

$x = 0 \quad \theta = 0$

$x = 2 \quad \theta = \pi/3$

$3x^2+4 = 3 \times \frac{4}{3}\tan^2\theta + 4$

$= 4\tan^2\theta + 4$

$= 4(1+\tan^2\theta)$

$= 4\sec^2\theta$

$$= \frac{2}{\sqrt{3}}\int_0^{\pi/3} \frac{\sec^2\theta}{(4\sec^2\theta)^{3/2}}\,d\theta$$

$$= \frac{2}{\sqrt{3}}\int_0^{\pi/3} \frac{\sec^2\theta}{8\sec^3\theta}\,d\theta$$

$$= \frac{1}{4\sqrt{3}}\int_0^{\pi/3} \cos\theta\, d\theta$$

$$= \frac{1}{4\sqrt{3}}\Big[\sin\theta\Big]_0^{\pi/3} = \frac{1}{4\sqrt{3}}\left(\frac{\sqrt{3}}{2} - 0\right) = \frac{1}{8}$$