Question 282 (****+)

The height of tide, h meters, in a harbour on a certain day can be modelled by

$$h(t) = 10 + \sqrt{3}\sin(30t)^\circ + \cos(30t)^\circ,\ 0 \le t \le 12,$$

where t is the time in hours since midnight.

a) Find the time when the high tide and the low tide occur during the morning hours of that day and state the corresponding depth of water in the harbour at these times.

The depth of water in this harbour needs to be at least 8.5 metres for a boat to dock

A boat arrives outside the harbour at high tide and needs five hours to unload.

b) Show that the boat has to wait until $09:23$ to enter the harbour.

$\boxed{\text{high tide of 12 metres at 02:00}}$, $\boxed{\text{low tide of 8 metres at 08:00}}$

(a) $h(t) = 10 + \sqrt{3}\sin(30t) + \cos(30t)$ OR $R\sin(30t+\alpha)$

$\sqrt{3}\sin(30t) + \cos(30t) \equiv R\cos(30t - \alpha)$

$\equiv R\cos 30t \cos\alpha + R\sin 30t \sin\alpha$

$\equiv (R\cos\alpha)\cos 30t + (R\sin\alpha)\sin 30t$

$R\cos\alpha = 1$
$R\sin\alpha = \sqrt{3}$

$R = \sqrt{1^2 + \sqrt{3}^2} = 2$

$\tan\alpha = \sqrt{3} \implies \alpha = 60°$

$\therefore \boxed{h(t) = 10 + 2\cos(30t - 60°)}$ (or $h(t) = 10 + 2\sin(30t + 30)$)

HIGH TIDE IS 12 metres $\implies \cos(30t - 60) = 1$

$30t - 60 = 0$

$30t = 60$

$t = 2$

LOW TIDE IS 8 METRES $\implies \cos(30t - 60) = -1$

$30t - 60 = 180$

$30t = 240$

$t = 8$

$\therefore$ HIGH TIDE 12 metres AT 02:00
LOW TIDE 8 metres AT 08:00

(b) $h = 8.5$

$\implies 8.5 = 10 + 2\cos(30t - 60)$

$\implies -1.5 = 2\cos(30t - 60)$

$\implies \cos(30t - 60) = -0.75$

$\implies \begin{cases} 30t - 60 = 138.59 \pm 360n \\ 30t - 60 = 221.41 \pm 360n \end{cases} \quad n = 0,1,2,3,\ldots$

$\begin{cases} 30t = 198.59 \pm 360n \\ 30t = 281.41 \pm 360n \end{cases}$

$\begin{cases} t = 6.62 \pm 12n \\ t = 9.38 \pm 12n \end{cases}$

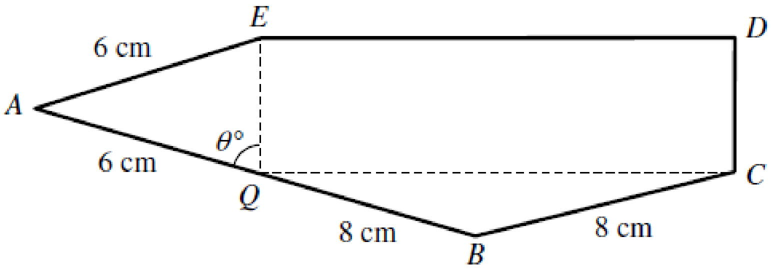
- SHIP ARRIVES AT 02:00
  + 5 HOURS
  07:00

IT CANNOT GO IN BECAUSE WHEN $t = 6.62$ (06:37) IT RUNS AROUND

$\therefore$ IT MUST WAIT UNTIL
$t = 9.38$
ie $0.38 \times 60 = 22.8$
ie 09:23

AS REQUIRED

Question 283 (****+)



The figure above shows an irregular pentagon $ABCDE$. The lengths of AB, BC and AE are 14 cm, 8 cm and 6 cm respectively.

The point Q lies on AB so that AQ is 6 cm and QB is 8 cm. The point D is then constructed so that $QEDC$ is a rectangle.

Let the angle AQE be $\theta°$ and assume that $\theta°$ can vary.

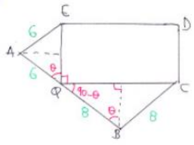
a) Given that P cm and R cm^2 are the perimeter and the area of the pentagon respectively, show that …

- **i.** … $P = 28 + 12\cos\theta + 16\sin\theta$.
- **ii.** … $R = 146\sin 2\theta$.

b) Hence show that when the pentagon has a maximum area

$$P = 14\left(2+\sqrt{2}\right)\ \text{cm}^2.$$

(a)



- $|EQ| = 2|AQ|\cos\theta$
  $= 2 \times 6 \times \cos\theta$
  $= 12\cos\theta$

  $\therefore |DC| = |EQ| = 12\cos\theta$

- $|QC| = 2|QB|\sin\theta$
  $= 2 \times 8\sin\theta$
  $= 16\sin\theta$

  $\therefore |ED| = |QC| = 16\sin\theta$

$\therefore$ PERIMETER $= 6 + 6 + 8 + 8 + 12\cos\theta + 16\sin\theta$

$= 28 + 12\cos\theta + 16\sin\theta$

AS REQUIRED

(b) AREA $= \frac{1}{2}|AQ||EQ|\sin\theta + \frac{1}{2}|QB||BC|\sin 2\theta + |EQ||QC|$

$= \frac{1}{2} \times 6 \times 12\cos\theta\sin\theta + \frac{1}{2} \times 8 \times 8\sin 2\theta + 12\cos\theta \times 16\sin\theta$

$= 36\cos\theta\sin\theta + 32\sin 2\theta + 192\sin\theta\cos\theta$

$= 228\cos\theta\sin\theta + 32\sin 2\theta \quad = 114(2\sin\theta\cos\theta) + 32\sin 2\theta$

$= 114\sin 2\theta + 32\sin 2\theta$

$= 146\sin 2\theta$

AS REQUIRED

(c) MAX AREA $= 146$ (OCCURS WHEN $\sin 2\theta = 1$

$2\theta = \pi/2 \pm 2n\pi \quad n = 0, 1, 2, 3 \ldots$

$\theta = \pi/4 \pm n\pi$

$\therefore \theta = \frac{\pi}{4}$

$\therefore \text{PERIMETER}_{\text{MAX}} = 28 + 12\cos\frac{\pi}{4} + 16\sin\frac{\pi}{4}$

$= 28 + 6\sqrt{2} + 8\sqrt{2}$

$= 28 + 14\sqrt{2}$

$= 14(2 + \sqrt{2})$

AS REQUIRED