To prove that the derivative of $\sin x$ is $\cos x$

Let $f(x) = \sin x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin x (\cos h - 1)}{h} + \cos x \left(\frac{\sin h}{h}\right) \right]$$

As $h \to 0$, $\cos h \to 1$ $\therefore \lim_{h \to 0} \left(\frac{\cos h - 1}{h}\right) \doteq 0$

As $h \to 0$, $\frac{\sin h}{h} \to 1$ $\therefore \lim_{h \to 0} \frac{\sin h}{h} = 1$

$$\therefore f'(x) = (\sin x)(0) + (\cos x)(1)$$

$$= \cos x$$

Hence the derivative of $\sin x$ is $\cos x$

To prove that the derivative of $\cos x$ is $-\sin x$

Let $f(x) = \cos x$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \lim_{h\to 0} \left[\cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right) \right]$$

As $h \to 0$, $\cos h \to 1$ $\quad \therefore \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) = 0$

As $h \to 0$, $\frac{\sin h}{h} \to 1$ $\quad \therefore \lim_{h\to 0} \left(\frac{\sin h}{h}\right) = 1$

$$\therefore f'(x) = (\cos x)(0) - (\sin x)(1)$$

$$= -\sin x$$