Question 1 (**)

The gradient of a curve satisfies

$$\frac{dy}{dx} = \frac{1}{3y^2(x-1)}, \quad x > 1.$$

Given the curve passes though the point $P(2,-1)$ and the point $Q(q,1)$, determine the exact value of q.

$\boxed{q = 1 + e^2}$

Question 2 (**+)

Water is draining out of a tank so that the height of the water, h m, in time t minutes, satisfies the differential equation

$$\frac{dh}{dt} = -k\sqrt{h} ,$$

where k is a positive constant.

The initial height of the water is 2.25 m and 20 minutes later it drops to 1 m.

a) Show that the solution of the differential equation can be written as

$$h = \frac{(60-t)^2}{1600} .$$

b) Find after how long the height of the water drops to 0.25 m.

$\boxed{t = 40}$

Question 3 (**+)

The radius, r mm, of a circular ink stain, t seconds after it was formed, satisfies the differential equation

$$\frac{dr}{dt} = \frac{k}{r}, \; r \neq 0$$

where k is a positive constant.

The initial radius of the stain is 4 mm and 8 seconds later it has increased to 20 mm.

a) Solve the differential equation to show that

$$r = 4\sqrt{3t+1}.$$

b) Find the time when the stain will have a **circumference** of 56π.

c) Explain why this model is only likely to hold for small values of t.

$\boxed{t = 16}$

Question 4 (**+)

An entomologist believes that the population P insects in a colony, t weeks after it was first observed, obeys the differential equation

$$\frac{dP}{dt} = kP^2,$$

where k is a positive constant.

Initially 1000 insects were observed, and this population doubled after 4 weeks.

a) Find a solution of the differential equation, in the form $P = f(t)$.

b) Give <u>two different</u> reasons why the model can only work for small values of t.

$$\boxed{P = \frac{8000}{8-t}}$$

Question 5 (**+)

The area, A km^2, of an oil spillage on the surface of the sea, at time t hours after it was formed, satisfies the differential equation

$$\frac{dA}{dt} = \frac{A^{\frac{3}{2}}}{t^2}, \quad t > 0.$$

When $t = 1$, $A = 0.25$.

a) Find a solution of the differential equation, in the form $A = f(t)$.

b) Determine the largest area that the oil spillage will ever attain.

$\boxed{A = \frac{4t^2}{(3t+1)^2}}$, $\boxed{A_{max} \to \frac{4}{9}}$

Question 6 (***)

The mass, m grams, of a burning candle, t hours after it was lit up, satisfies the differential equation

$$\frac{dm}{dt} = -k(m-10),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$m = 10 + A\mathrm{e}^{-kt},$$

where A is a non-zero constant.

The initial mass of the candle was 120 grams, and 3 hours later its mass has halved.

b) Find the value of A and show further that

$$k = \frac{1}{3}\ln\left(\frac{11}{5}\right).$$

c) Calculate, correct to three significant figures, the mass of the candle after a further period of 3 hours has elapsed.

$\boxed{}$, $\boxed{A = 110}$, $\boxed{m = 32.7}$

Question 7 (***)

A radioactive isotope decays in such a way so that the number N of the radioactive nuclei present at time t days, satisfies the differential equation

$$\frac{dN}{dt} = -kN,$$

where k is a positive constant.

a) Show clearly that

$$N = A\mathrm{e}^{-kt},$$

where A is a non zero constant.

Initially there were 6.00×10^{24} radioactive nuclei and 10 days later this number reduced to 6.25×10^{22}.

b) Show further that $k = 0.45643$, correct to five decimal places.

c) Calculate the number of the radioactive nuclei after a further period of 10 days has elapsed.

$\boxed{6.51 \times 10^{20}}$

Question 8 (***)

The number of fish x in a small lake at time t months after a certain instant, is modelled by the differential equation

$$\frac{dx}{dt} = x(1-kt),$$

where k is a positive constant.

We may assume that x can be treated as a continuous variable.

It is estimated that there are 10000 fish in the lake when $t=0$ and 12 months later the number of fish returns back to 10000.

a) Find a solution of the differential equation, in the form $x = f(t)$.

b) Find the long term prospects for this population of fish.

$\boxed{x = 10000e^{t-\frac{1}{12}t^2}}$, $\boxed{x \to 0}$

Question 9 (***)

The area, A km^2, of an oil spillage is growing in time t hours according the differential equation

$$\frac{dA}{dt} = \frac{4e^t}{\sqrt{A}}, \quad A > 0.$$

The initial area of the oil spillage was 4 km^2.

a) Solve the differential equation to show that

$$A^3 = 4\left(3e^t + 1\right)^2.$$

b) Find, to three significant figures, the value of t when the area of the spillage reaches 1000 km^2.

$\boxed{t = 8.57}$

Question 10 (***+)

A cylindrical tank of height 150 cm is full of oil which started leaking out from a small hole at the side of a tank.

Let h cm be the height of the oil still left in the tank, after leaking for t minutes, and assume the leaking can be modelled by the differential equation

$$\frac{dh}{dt} = -\frac{1}{4}(h-6)^{\frac{3}{2}}.$$

a) Solve the differential equation to show that …

i. … $t = \frac{8}{\sqrt{h-6}} - \frac{2}{3}$.

ii. … $\sqrt{h-6} = \frac{24}{3t+2}$.

b) State how high is the hole from the bottom of the tank and hence show further that it takes 200 seconds for the oil level to reach 4 cm above the level of the hole.

6cm

Question 11 (***+)

Water is leaking out of a hole at the side of a tank.

Let the height of the water in the tank is y cm at time t minutes.

The rate at which the height of the water in the tank is decreasing is modelled by the differential equation

$$\frac{dy}{dt} = -6(y-7)^{\frac{2}{3}}.$$

When $t = 0$, $y = 132$.

a) Find how long it takes for the water level to drop from 132 cm to 34 cm.

The tank is filled up with water again to a height of 132 cm and allowed to leak out in exactly the same fashion as the one described in part (**a**).

b) Determine how long it takes for the water to stop leaking.

$\boxed{}$, $\boxed{t=1}$, $\boxed{t=2.5}$

Question 12 (***+)

The speed, v ms^{-1}, of a skydiver falling through still air t seconds after jumping off a plane, can be modelled by the differential equation

$$8\frac{dv}{dt} = 80 - v.$$

The skydiver jumps off the plane with a downward speed of 5 ms^{-1}.

a) Solve the differential equation to show that

$$v = 80 - 75e^{-\frac{1}{8}t}.$$

b) Find the maximum possible speed that the skydiver can achieve and show that this speed is independent of the speed he jumps off the plane

You may assume that the skydiver cannot possible jump at a speed greater than his subsequent maximum speed.

$\boxed{80\,\text{ms}^{-1}}$

Question 13 (***+)

A population P, in millions, at a given time t years, satisfies the differential equation

$$\frac{dP}{dt} = P(1-P).$$

Initially the population is one quarter of a million.

a) Solve the differential equation to show that

$$\frac{3P}{1-P} = e^{t}.$$

b) Show further that

$$P = \frac{1}{1+3e^{-t}}.$$

c) Show mathematically that the limiting value for this population is one million.

d) Find, to two decimal places, the time it takes for the population to reach three quarters of its limiting value.

$\boxed{}$, $\boxed{t \to \infty, \quad P \to 1}$, $\boxed{t = \ln 9 = 2.20}$

Question 14 (***+)

The number of foxes N, in thousands, living within an urban area t years after a given instant, can be modelled by the differential equation

$$\frac{dN}{dt} = 2N - N^2, \quad t > 0.$$

Initially it is thought 1000 foxes lived within this urban area.

a) Find a solution of the differential equation, in the form $N = f(t)$.

b) Find the long term prospects of this population of foxes, as predicted by this model, clearly showing your reasoning.

$\boxed{}$, $\boxed{N = \frac{2}{1+e^{-2t}} \text{ or } N = \frac{2e^{2t}}{e^{2t}+1}}$, $\boxed{\text{population} \to 2000}$

Question 15 (***+)

A machine is used to produce waves in the swimming pool of a water theme park.

Let x cm be the height of the wave produced above a certain level in the pool, and suppose it can be modelled by the differential equation

$$\frac{dx}{dt} = 2x\sin 2t, \quad t > 0,$$

where t is the time in seconds.

When $t = 0$, $x = 6$.

a) Solve the differential equation to show

$$x = 6e^{1-\cos 2t}.$$

b) Find the maximum height of the wave.

$\boxed{}$, $\boxed{x_{max} = 44.3 \text{ cm}}$

Question 16 (***+)

Food is placed in a preheated oven maintained at a constant temperature of $200\ ^\circ\mathrm{C}$.

Let $\theta\ ^\circ\mathrm{C}$ be temperature of the food t minutes after it was placed in the oven.

It is assumed that θ satisfies the differential equation

$$\frac{d\theta}{dt} = k(200-\theta),$$

where k is a positive constant.

a) Solve the differential equation to show that

$$\theta = 200 + A\mathrm{e}^{-kt},$$

where A is a non zero constant.

When a food item was placed in this oven it had a temperature of $20\ ^\circ\mathrm{C}$ and 10 minutes later its temperature had risen to $120\ ^\circ\mathrm{C}$.

b) Show further that $k = 0.0811$.

c) Find the value of t when the food item reaches a temperature of $160\ ^\circ\mathrm{C}$.

$\boxed{}$, $\boxed{t = 18.55}$

Question 17 (****)

$$\frac{1}{t\left(t^2+1\right)} \equiv \frac{At+B}{t^2+1} + \frac{C}{t}.$$

a) Find the value of each of the constants A, B and C.

In a chemical reaction, the mass m grams of the chemical produced at time t minutes, satisfies the differential equation

$$\frac{dm}{dt} = \frac{m}{t\left(t^2+1\right)}.$$

b) Find a general solution of the differential equation, in the form $m = f(t)$.

Two minutes after the reaction started the mass produced is 10 grams.

c) Calculate how many grams will be produced after 4 minutes.

d) Determine, in exact surd form, the maximum mass that will ever be produced by this chemical reaction.

$\boxed{A=-1,\ B=0,\ C=1}$, $\boxed{m = \frac{kt}{\sqrt{t^2+1}}}$, $\boxed{\frac{20}{17}\sqrt{85} \approx 10.85}$, $\boxed{m_{\text{max}} = 5\sqrt{5}}$

Question 18 (****)

The population of a herd of zebra, P thousands, in time t years is thought to be governed by the differential equation

$$\frac{dP}{dt} = \frac{1}{20} P(2P-1)\cos t .$$

It is assumed that since P is large it can be modelled as a continuous variable, and its initial value is 8.

a) Solve the differential equation to show that

$$P = \frac{8}{16 - 15e^{\frac{1}{20}\sin t}} .$$

b) Find the maximum and minimum population of the herd.

$\boxed{}$, $\boxed{P_{max} = 34642}$, $\boxed{P_{min} = 4620}$

Question 19 (****)

A population p, in millions, is thought to obey the differential equation

$$\frac{dp}{dt} = kp\cos kt$$

where k is a positive constant, and t is measured in days from a certain instant.

When $t = 0$, $p = p_0$.

a) Solve the differential equation to find p in terms of p_0, k and t.

The value of k is now assumed to be 3.

b) Find, to the nearest minute, the time for the population to reach p_0 again, for the first time.

$\boxed{p = p_0 e^{\sin kt}}$, $\boxed{1508 \text{ min}}$

Question 20 (****)

Cars are attached to a giant wheel on a fairground ride, and they can be made to lower or rise in height as the wheel is turning around.

Let the height above ground of one such car be h metres, and let t be the time in seconds, since the ride starts.

It may be assumed that h satisfies the differential equation

$$\frac{dh}{dt} = \frac{3}{2}\sqrt{h}\sin\left(\frac{3t}{4}\right).$$

a) Solve the differential equation subject to the condition $t = 0$, $h = 1$, to show

$$\sqrt{h} = 2 - \cos\left(\frac{3t}{4}\right).$$

b) Find the greatest height of the car above ground.

c) Find the value of t when the car reaches a height of 8 metres above the ground <u>for the third time</u>, since the ride started.

$\boxed{h_{max} = 9}$, $\boxed{t = 11.77}$