PARTIAL FRACTIONS

1 Given that

$$\frac{22}{(2x-3)(x+4)} \equiv \frac{A}{2x-3} + \frac{B}{x+4},$$

find the values of the constants A and B.

2 Find the values of A, B and C such that

$$\frac{x+5}{(x+1)(x-3)^2} \equiv \frac{A}{x+1} + \frac{B}{x-3} + \frac{C}{(x-3)^2}.$$

3 Given that

$$\frac{4x^2-16x-7}{2x^2-9x+4} \equiv A + \frac{B}{2x-1} + \frac{C}{x-4},$$

find the values of the constants A, B and C.

4 The function f is defined by

$$f(x) = \frac{4}{x^2-1}.$$

a Express f(x) in partial fractions.

The function g is defined by

$$g(x) = \frac{2+5x-x^2}{(x-4)(x-2)(x-1)}.$$

b Express g(x) in partial fractions.

c Hence, or otherwise, solve the equation $f(x) = g(x)$.

5 **a** Express $\frac{x-2}{(1-x)(1-2x)}$ in partial fractions.

b Hence find the series expansion of $\frac{x-2}{(1-x)(1-2x)}$ in ascending powers of x up to and including the term in x^3 and state the set of values of x for which the expansion is valid.

6

$$f(x) \equiv \frac{4-17x}{(1+2x)(1-3x)^2}, \quad |x| < \tfrac{1}{3}.$$

a Express f(x) in partial fractions.

b Hence, or otherwise, find the series expansion of f(x) in ascending powers of x up to and including the term in x^3.

PARTIAL FRACTIONS — *Answers*

1 $22 \equiv A(x+4) + B(2x-3)$

$x = \frac{3}{2} \quad \Rightarrow \quad 22 = \frac{11}{2}A \quad \Rightarrow \quad A = 4$

$x = -4 \quad \Rightarrow \quad 22 = -11B \quad \Rightarrow \quad B = -2$

2 $x + 5 \equiv A(x-3)^2 + B(x+1)(x-3) + C(x+1)$

$x = -1 \quad \Rightarrow \quad 4 = 16A \quad \Rightarrow \quad A = \frac{1}{4}$

$x = 3 \quad \Rightarrow \quad 8 = 4C \quad \Rightarrow \quad C = 2$

coeffs of $x^2 \quad \Rightarrow \quad 0 = A + B \quad \Rightarrow \quad B = -\frac{1}{4}$

3 $4x^2 - 16x - 7 \equiv A(2x-1)(x-4) + B(x-4) + C(2x-1)$

$x = 4 \quad \Rightarrow \quad -7 = 7C \quad \Rightarrow \quad C = -1$

$x = \frac{1}{2} \quad \Rightarrow \quad -14 = -\frac{7}{2}B \quad \Rightarrow \quad B = 4$

coeffs of $x^2 \quad \Rightarrow \quad 4 = 2A \quad \Rightarrow \quad A = 2$

4 **a** $\frac{4}{(x+1)(x-1)} \equiv \frac{A}{x+1} + \frac{B}{x-1}$

$4 \equiv A(x-1) + B(x+1)$

$x = -1 \quad \Rightarrow \quad 4 = -2A \quad \Rightarrow \quad A = -2$

$x = 1 \quad \Rightarrow \quad 4 = 2B \quad \Rightarrow \quad B = 2$

$\therefore \ \mathrm{f}(x) \equiv \frac{2}{x-1} - \frac{2}{x+1}$

b $\frac{2+5x-x^2}{(x-4)(x-2)(x-1)} \equiv \frac{A}{x-4} + \frac{B}{x-2} + \frac{C}{x-1}$

$2 + 5x - x^2 \equiv A(x-2)(x-1) + B(x-4)(x-1) + C(x-4)(x-2)$

$x = 4 \quad \Rightarrow \quad 6 = 6A \quad \Rightarrow \quad A = 1$

$x = 2 \quad \Rightarrow \quad 8 = -2B \quad \Rightarrow \quad B = -4$

$x = 1 \quad \Rightarrow \quad 6 = 3C \quad \Rightarrow \quad C = 2$

$\therefore \ \mathrm{g}(x) \equiv \frac{1}{x-4} - \frac{4}{x-2} + \frac{2}{x-1}$

c $\frac{2}{x-1} - \frac{2}{x+1} = \frac{1}{x-4} - \frac{4}{x-2} + \frac{2}{x-1}$

$\frac{4}{x-2} - \frac{1}{x-4} - \frac{2}{x+1} = 0$

$\frac{4(x-4)(x+1) - (x-2)(x+1) - 2(x-2)(x-4)}{(x-2)(x-4)(x+1)} = 0$

$4(x^2 - 3x - 4) - (x^2 - x - 2) - 2(x^2 - 6x + 8) = 0$

$x^2 + x - 30 = 0$

$(x+6)(x-5) = 0$

$x = -6, 5$

5 **a** $\dfrac{x-2}{(1-x)(1-2x)} \equiv \dfrac{A}{1-x} + \dfrac{B}{1-2x}$

$x-2 \equiv A(1-2x) + B(1-x)$

$x = 1 \quad \Rightarrow \quad -1 = -A \quad \Rightarrow \quad A = 1$

$x = \frac{1}{2} \quad \Rightarrow \quad -\frac{3}{2} = \frac{1}{2}B \quad \Rightarrow \quad B = -3$

$\therefore \ \dfrac{x-2}{(1-x)(1-2x)} \equiv \dfrac{1}{1-x} - \dfrac{3}{1-2x}$

b $\dfrac{1}{1-x} = (1-x)^{-1} = 1 + (-1)(-x) + \frac{(-1)(-2)}{2}(-x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-x)^3 + \ldots$

$= 1 + x + x^2 + x^3 + \ldots, \ |-x| < 1 \ \therefore |x| < 1$

$\dfrac{3}{1-2x} = 3(1-2x)^{-1} = 3[1 + (-1)(-2x) + \frac{(-1)(-2)}{2}(-2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-2x)^3 + \ldots]$

$= 3 + 6x + 12x^2 + 24x^3 + \ldots, \ |-2x| < 1 \ \therefore \ |x| < \frac{1}{2}$

$\therefore \ \dfrac{x-2}{(1-x)(1-2x)} = (1 + x + x^2 + x^3 + \ldots) - (3 + 6x + 12x^2 + 24x^3 + \ldots)$

$= -2 - 5x - 11x^2 - 23x^3 + \ldots,$ valid for $|x| < \frac{1}{2}$

6 **a** $\dfrac{4-17x}{(1+2x)(1-3x)^2} \equiv \dfrac{A}{1+2x} + \dfrac{B}{1-3x} + \dfrac{C}{(1-3x)^2}$

$4 - 17x \equiv A(1-3x)^2 + B(1+2x)(1-3x) + C(1+2x)$

$x = -\frac{1}{2} \quad \Rightarrow \quad \frac{25}{2} = \frac{25}{4}A \quad \Rightarrow \quad A = 2$

$x = \frac{1}{3} \quad \Rightarrow \quad -\frac{5}{3} = \frac{5}{3}C \quad \Rightarrow \quad C = -1$

coeffs of $x^2 \ \Rightarrow \quad 0 = 9A - 6B \quad \Rightarrow \quad B = 3$

$\therefore \ f(x) \equiv \dfrac{2}{1+2x} + \dfrac{3}{1-3x} - \dfrac{1}{(1-3x)^2}$

b $\dfrac{2}{1+2x} = 2(1+2x)^{-1} = 2[1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(2x)^3 + \ldots]$

$= 2 - 4x + 8x^2 - 16x^3 + \ldots$

$\dfrac{3}{1-3x} = 3(1-3x)^{-1} = 3[1 + (-1)(-3x) + \frac{(-1)(-2)}{2}(-3x)^2 + \frac{(-1)(-2)(-3)}{3\times 2}(-3x)^3 + \ldots]$

$= 3 + 9x + 27x^2 + 81x^3 + \ldots$

$\dfrac{1}{(1-3x)^2} = (1-3x)^{-2} = 1 + (-2)(-3x) + \frac{(-2)(-3)}{2}(-3x)^2 + \frac{(-2)(-3)(-4)}{3\times 2}(-3x)^3 + \ldots$

$= 1 + 6x + 27x^2 + 108x^3 + \ldots$

$f(x) = (2 - 4x + 8x^2 - 16x^3 + \ldots) + (3 + 9x + 27x^2 + 81x^3 + \ldots) - (1 + 6x + 27x^2 + 108x^3 + \ldots)$

$= 4 - x + 8x^2 - 43x^3 + \ldots$