BINOMIAL EXPANSION

1 **a** Expand $(1-2x)^{\frac{1}{2}}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

b By substituting a suitable value of x in your expansion, find an estimate for $\sqrt{0.98}$

c Show that $\sqrt{0.98} = \frac{7}{10}\sqrt{2}$ and hence find the value of $\sqrt{2}$ correct to 8 significant figures.

2 **a** Expand $(1+2x)^{-1}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

b Hence find the series expansion of $\frac{1-x}{1+2x}$, $|x| < \frac{1}{2}$, in ascending powers of x up to and including the term in x^3.

3 **a** Expand $(9-6x)^{\frac{1}{2}}$, $|x| < \frac{3}{2}$, in ascending powers of x up to and including the term in x^3, simplifying the coefficient in each term.

b Use your expansion with a suitable value of x to find the value of $\sqrt{8.7}$ correct to 7 significant figures.

4 Find the coefficient of x^2 in the series expansion of $\frac{2+x}{\sqrt{4-2x}}$, $|x| < 2$.

5 The first three terms in the expansion of $(1+ax)^b$, in ascending powers of x, for $|ax| < 1$, are

$$1 - 6x + 24x^2.$$

a Find the values of the constants a and b.

b Find the coefficient of x^3 in the expansion.

6 **a** Expand $(2-x)^{-2}$, $|x| < 2$, in ascending powers of x up to and including the term in x^3.

b Hence, find the coefficient of x^3 in the series expansion of $\frac{3-x}{(2-x)^2}$.

7

$$f(x) \equiv \frac{4}{\sqrt{1+\frac{2}{3}x}}, \quad -\frac{3}{2} < x < \frac{3}{2}.$$

a Show that $f(\frac{1}{10}) = \sqrt{15}$.

b Expand $f(x)$ in ascending powers of x up to and including the term in x^2.

c Use your expansion to obtain an approximation for $\sqrt{15}$, giving your answer as an exact, simplified fraction.

d Show that $3\frac{55}{63}$ is a more accurate approximation for $\sqrt{15}$.

8 **a** Find the binomial expansion of $(4+x)^{\frac{1}{2}}$ in ascending powers of x up to and including the term in x^2 and state the set of values of x for which the expansion is valid.

b By substituting $x = \frac{1}{20}$ in your expansion, find an estimate for $\sqrt{5}$, giving your answer to 9 significant figures.

c Obtain the value of $\sqrt{5}$ from your calculator and hence comment on the accuracy of the estimate found in part **b**.

1

a $= 1 + (\frac{1}{2})(-2x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-2x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-2x)^3 + \ldots$

$= 1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \ldots$

b $\sqrt{0.98} = (1 - 2x)^{\frac{1}{2}}$ when $x = 0.01$

$\therefore\ \sqrt{0.98} \approx 1 - (0.01) - \frac{1}{2}(0.01)^2 - \frac{1}{2}(0.01)^3$

$= 1 - 0.01 - 0.000\,05 - 0.000\,000\,5$

$= 0.989\,949\,5$

c $\sqrt{0.98} = \sqrt{\frac{98}{100}} = \sqrt{\frac{49\times2}{100}} = \frac{7}{10}\sqrt{2}$

$\therefore\ \sqrt{2} \approx \frac{10}{7} \times 0.989\,949\,5 = 1.414\,213\,6$ (8sf)

2

a $= 1 + (-1)(2x) + \frac{(-1)(-2)}{2}(2x)^2 + \frac{(-1)(-2)(-3)}{3\times2}(2x)^3 + \ldots$

$= 1 - 2x + 4x^2 - 8x^3 + \ldots$

b $= (1 - x)(1 + 2x)^{-1} = (1 - x)(1 - 2x + 4x^2 - 8x^3 + \ldots)$

$= 1 - 2x + 4x^2 - 8x^3 - x + 2x^2 - 4x^3 + \ldots$

$= 1 - 3x + 6x^2 - 12x^3 + \ldots$

3

a $= 9^{\frac{1}{2}}(1 - \frac{2}{3}x)^{\frac{1}{2}} = 3(1 - \frac{2}{3}x)^{\frac{1}{2}}$

$= 3[1 + (\frac{1}{2})(-\frac{2}{3}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(-\frac{2}{3}x)^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3\times2}(-\frac{2}{3}x)^3 + \ldots]$

$= 3 - x - \frac{1}{6}x^2 - \frac{1}{18}x^3 + \ldots$

b let $x = 0.05$

$\sqrt{8.7} \approx 3 - 0.05 - \frac{1}{6}(0.05)^2 - \frac{1}{18}(0.05)^3$

$= 2.949\,576$ (7sf)

4

$\frac{2+x}{\sqrt{4-2x}} = (2 + x)(4 - 2x)^{-\frac{1}{2}} = (2 + x) \times 4^{-\frac{1}{2}}(1 - \frac{1}{2}x)^{-\frac{1}{2}}$

$= (2 + x) \times \frac{1}{2}[1 + (-\frac{1}{2})(-\frac{1}{2}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(-\frac{1}{2}x)^2 + \ldots]$

$= (2 + x)(\frac{1}{2} + \frac{1}{8}x + \frac{3}{64}x^2 + \ldots)$

$\therefore$ coeff of $x^2 = (2 \times \frac{3}{64}) + (1 \times \frac{1}{8}) = \frac{7}{32}$

5

a $(1 + ax)^b = 1 + b(ax) + \frac{b(b-1)}{2}(ax)^2 + \ldots$

$\therefore\quad ab = -6 \quad (1)$

and $\quad \frac{1}{2}a^2b(b - 1) = 24 \quad (2)$

$(1) \quad \Rightarrow \quad a = -\frac{6}{b}$

sub. (2) $\Rightarrow \quad \frac{18}{b}(b - 1) = 24$

$18b - 18 = 24b$

$b = -3$

$a = 2$

b $= \frac{(-3)(-4)(-5)}{3\times2}(2)^3 = -80$

6

a $= 2^{-2}(1 - \frac{1}{2}x)^{-2} = \frac{1}{4}(1 - \frac{1}{2}x)^{-2}$

$= \frac{1}{4}[1 + (-2)(-\frac{1}{2}x) + \frac{(-2)(-3)}{2}(-\frac{1}{2}x)^2 + \frac{(-2)(-3)(-4)}{3 \times 2}(-\frac{1}{2}x)^3 + \ldots]$

$= \frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \ldots$

b $\frac{3-x}{(2-x)^2} = (3-x)(2-x)^{-2} = (3-x)(\frac{1}{4} + \frac{1}{4}x + \frac{3}{16}x^2 + \frac{1}{8}x^3 + \ldots)$

$\therefore$ coefficient of $x^3 = (3 \times \frac{1}{8}) + (-1 \times \frac{3}{16}) = \frac{3}{16}$

7

a $f(\frac{1}{10}) = \frac{4}{\sqrt{1+\frac{1}{15}}} = \frac{4}{\sqrt{\frac{16}{15}}} = \frac{4}{\frac{4}{\sqrt{15}}} = 4 \times \frac{\sqrt{15}}{4} = \sqrt{15}$

b $= 4(1 + \frac{2}{3}x)^{-\frac{1}{2}} = 4[1 + (-\frac{1}{2})(\frac{2}{3}x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(\frac{2}{3}x)^2 + \ldots]$

$= 4 - \frac{4}{3}x + \frac{2}{3}x^2 + \ldots$

c $\sqrt{15} = f(\frac{1}{10}) \approx 4 - \frac{4}{3} \times \frac{1}{10} + \frac{2}{3} \times (\frac{1}{10})^2 + \ldots$

$= 4 - \frac{2}{15} + \frac{1}{150} = 3\frac{131}{150}$

d $\sqrt{15} = 3.872\,98\ldots$

$3\frac{131}{150} = 3.873\,33\ldots$

$3\frac{55}{63} = 3.873\,01\ldots$

$\therefore \sqrt{15} < 3\frac{55}{63} < 3\frac{131}{150}$, so $3\frac{55}{63}$ is a more accurate approximation

8

a $= 4^{\frac{1}{2}}(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2(1 + \frac{1}{4}x)^{\frac{1}{2}} = 2[1 + (\frac{1}{2})(\frac{1}{4}x) + \frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\frac{1}{4}x)^2 + \ldots]$

$= 2 + \frac{1}{4}x - \frac{1}{64}x^2 + \ldots$, $|\frac{1}{4}x| < 1$ $\therefore$ valid for $|x| < 4$

b when $x = \frac{1}{20}$, $(4 + x)^{\frac{1}{2}} \approx 2 + \frac{1}{4}(\frac{1}{20}) - \frac{1}{64}(\frac{1}{20})^2$

$= 2.012\,460\,938$

$(4 + \frac{1}{20})^{\frac{1}{2}} = \sqrt{\frac{81}{20}} = \sqrt{\frac{81 \times 5}{100}} = \frac{9}{10}\sqrt{5}$

$\therefore \sqrt{5} \approx \frac{10}{9} \times 2.012\,460\,938 = 2.236\,067\,71$ (9sf)

c $\sqrt{5} = 2.236\,067\,977\ldots$

$\therefore$ estimate is accurate to 7 significant figures